Optimal Consumption and Portfolio of Investment in a Financial Market with Jumps

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ABSTRACT

We will address the problems that in order to obtain a maximum expected utility from both consumption and terminal wealth, how should a small investor, choose his securities portfolio and his consumption rate at every time. We discuss these in a financial market with jumps, with Lagrangian methods taking into account to find out the possible optimal solutions. Then we prove they are really optimal. Finally, under the assumptions of deterministic coefficients on the model, we obtain the optimal pair of portfolio/consumption in an explicit feedback form on the current level of wealth.

KEYWORDS: Optimal Consumption and Portfolio, Lagrangian method, Dynamical Programming

INTRODUCTION

Lagrangian and dynamical programming methods are very popular in financial studies to solve optimal problems. Here we discuss the problems of maximization of utility from consumption, maximization of utility from terminal wealth and maximization of utility from both consumption and terminal wealth in a financial market with jumps. And we use mathematics very carefully.

Although direct Lagrangian procedure can deliver almost all the optimal solutions that the popular indirect-utilities method can deliver, and seems that it will do so more economically in the sense that redundant laborious calculations of any indirect utilities can be avoided, the solutions it delivers are only possible solutions. We still need to prove whether the possible solutions are real optimal or not. And the proving procedure is very complicating, whereas can not be avoided.

All the methods we use here can be extended to the problems of optimal investment, consumption and life insurance.

The terminology of financial market with jumps in this paper was defined in my paper last year, which was published in the proceeding of DSI 2014. We should pay attention to the items of Price of a stock or a bond, wealth process, consumption process, etc, ...

Utility functions

Let $U : (0, \infty) \to \mathbb{R}$ be a strictly increasing, strictly concave and $C^1$ function with $U(0) \triangleq \lim_{c \to 0^+} U(c) \geq -\infty$, $U'(\infty) \triangleq \lim_{c \to \infty} U'(c) = 0$. We allow the possibility that $U'(0) \triangleq \lim_{c \to 0^+} U'(c) = \infty$ and $U(0) = -\infty$. A function with these properties will be called a utility function.

The strict increase of $U(c)$ says that the investor prefers higher levels of consumption and/or terminal wealth to lower levels. The strict concavity of $U(\cdot)$ says that he is also risk-averse, i.e., that his marginal utility $U'(c)$ is decreasing, and tends to zero as $c \to \infty$ (a “saturation effect”).
Since $U' : [0, \infty] \to [0, U'(0)]$ is strictly decreasing, it has a strictly decreasing inverse $I : [0, U'(0)] \to [0, \infty]$. We extend $I$ to be a continuous function on the entirety of $[0, \infty]$ by setting $I(y) \equiv 0$ for $U'(0) \leq y \leq \infty$. It is easily to verify that

$$U(I(y)) - yI(y) \geq U(c) - yc, \quad 0 \leq c < \infty, \quad 0 < y < \infty,$$

by strict concavity of $U(\cdot)$.

For some of our results in this paper, we shall need to impose the additional conditions $U$ is a $C^2$ function and $U''$ is nondecreasing on $(0, \infty)$. (2)

**Lemma 1.1.** If $f(x)$ is convex and strictly decreasing (or respectively, increasing) on $(0, \infty)$, then the inverse function $g$ of $f$ is convex (respectively, concave) and strictly decreasing (respectively, increasing) on $(f(\infty), f(0))$ (respectively, $(f(0), f(\infty))$).

**Proof.** We just prove the convexity. For any $y_1 < y < y_2 \leq f(0)$ (respectively, $f(0) \leq y_1 < y < y_2$), there exist $x_2 < x < x_1$ (respectively, $x_1 < x < x_2$) such that $y_1 = f(x_1), y = f(x), y_2 = f(x_2)$.

Because $f$ is convex, then

$$\frac{f(x_2) - f(x)}{x_2 - x} \leq \frac{f(x_1) - f(x)}{x_1 - x} \quad \text{(respectively,} \quad \frac{f(x_1) - f(x)}{x_1 - x} \leq \frac{f(x_2) - f(x)}{x_2 - x} \text{)}.
$$

For $g$ is the inverse function of $f$, $x_1 = g(y_1), x = g(y), x_2 = g(y_2)$, thus we have

$$\frac{y_2 - y}{g(y_2) - g(y)} \leq \frac{y_2 - y}{g(y_1) - g(y)} \quad \text{(respectively,} \quad \frac{y_1 - y}{g(y_1) - g(y)} \leq \frac{y_2 - y}{g(y_2) - g(y)} \text{)}.
$$

Multiply two sides of the previous inequality by $(g(y_1) - g(y))(g(y_2) - g(y)) (> 0)$, we have

$$\frac{g(y_1) - g(y)}{y_1 - y} \leq \frac{g(y_2) - g(y)}{y_2 - y} \quad \text{(respectively,} \quad \frac{g(y_2) - g(y)}{y_2 - y} \leq \frac{g(y_1) - g(y)}{y_1 - y} \text{)},
$$

so $g$ is convex (respectively, concave).

Under condition (2), $U'$ is convex, so the inverse function $I$ of $U'$ is also convex by Lemma 1.1. And $I$ is continuously differentiable on $(0, \infty) \setminus \{U'(0)\}$. In the case $U'(0) < \infty$, we assume that $I'(U'(0)) = 0$, then because $I'(y) = 0$ for every $y > U'(0)$, the identity

$$\frac{dU(I(y))}{dy} = U'(I(y)).I'(y) = yU''(y)$$

becomes valid on $(0, \infty)$; in particular, if we have the additional condition

$$I'(U'(0)) = \lim_{y \to U'(0)^{+}} I'(y)$$

then $I$ is continuously differentiable on $(0, \infty)$. Besides, the identity (3) is also valid on $(0, \infty)$ if $U'(0) = \infty$.

**Maximization of utility from consumption**

In this section we shall try to address the following question. In order to obtain a maximum expected utility from consumption, how should a small investor, endowed with initial wealth $x > 0$, choose his securities portfolio $\pi(t)$ and his consumption rate $C(t)$ from among
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admissible pairs \((\pi, C) \in \mathcal{A}(x)\) at every time?

Suppose that we have a measurable, \(\mathcal{F}_t\)-adapted and uniformly bounded discount process \(\{\gamma(t), 0 \leq t \leq T\}\) and a utility function \(U_1\). Let \(\Gamma(t) \triangleq e^{-\int_0^t \gamma(u)du}\). We try to maximize the expected discounted utility from consumption

\[
J_1(x; \pi, C) = \mathbb{E} \int_0^T \Gamma(s)U_1(C(s))\,ds
\]

over the subclass

\[
\mathcal{A}_1(x) \triangleq \left\{(\pi, C) \in \mathcal{A}(x) : \mathbb{E} \int_0^T \Gamma(s)U_1^-(C(s))\,ds < \infty \right\}
\]

\(J_1\) in (5) is well defined for every pair \((\pi, C) \in \mathcal{A}_1(x)\), and \(\mathcal{A}_1(x) = \mathcal{A}(x)\) if \(U_1(0) > -\infty\). We denote by

\[
V_1(x) \triangleq \sup_{(\pi, C) \in \mathcal{A}_1(x)} J_1(x; \pi, C)
\]

the value function of this problem, and \(V_1(x) = -\infty\) if \(\mathcal{A}_1(x) = \emptyset\) for some \(x\).

Because utility comes only from consumption, we may ignore the portfolio process \(\pi\) by considering only consumption process \(C\) in \(\mathcal{D}(x)\). The following proposition will show this.

**Proposition 2.1** [Karatzas, 1989]. For every \(x \geq 0\), we have \(V_1(x) = \sup_{(\pi, C) \in \mathcal{A}_1(x)} \mathbb{E} \int_0^T \Gamma(s)U_1(C(s))\,ds\), and in particular,

\[
V_1(0) = U_1(0)\mathbb{E} \int_0^T \Gamma(s)\,ds
\]

**Proof.** Define \(\Delta(t) \triangleq Z(t)\beta(t)[\Gamma(t)]^{-1} = \zeta(t)[\Gamma(t)]^{-1}\) for later use.

Take \((\pi, C) \in \mathcal{A}_1(x)\), and consider \(z \triangleq \mathbb{E} \int_0^T \beta(s)C(s)\,ds \leq x\). If \(z > 0\), define

\[
\hat{C}(t) \triangleq \left( \frac{x}{z} \right) C(t), \text{ then } \hat{C} \in \mathcal{D}(x) \text{ and there exists a portfolio } \hat{\pi} \text{ such that } (\hat{\pi}, \hat{C}) \in \mathcal{A}(x).
\]

\(U_1(C(t)) \leq U_1(\hat{C}(t))\) since \(U_1\) is strictly increasing, then \(U_1^- (C(t)) \geq U_1^- (\hat{C}(t))\). Thus \(\hat{C}\) satisfies the condition in (6) since \(C\) does. Consequently, \((\hat{\pi}, \hat{C}) \in \mathcal{A}_1(x)\). If \(z = 0\), then \(C(t) = 0\), a.e., a.s., and let \(\hat{C}(t) \triangleq x/\mathbb{E} \int_0^T \beta(s)\,ds\), \(0 \leq t \leq T\), which is a constant and belongs to \(\mathcal{D}(x)\). Again, there exists a portfolio \(\hat{\pi}\) such that \((\hat{\pi}, \hat{C}) \in \mathcal{A}(x), \hat{C}\) satisfying (6), then \((\hat{\pi}, \hat{C}) \in \mathcal{A}_1(x)\). In the above two cases, they both have the property \(J_1(x; \pi, C) \leq J_1(x; \hat{\pi}, \hat{C})\).

According to [Karatzas, 1989], if \(x = 0\), then \(C(t) \equiv 0\), a.e., a.s., and this leads to (8). \(\square\)

Now we need only to solve the optimal consumption problem

\[
V_1(x) = \sup_{(\pi, C) \in \mathcal{A}(x)} \mathbb{E} \int_0^T \Gamma(s) \cdot U_1(C(s))\,ds
\]

when the constraint is given by

\[
\mathbb{E} \int_0^T \zeta(t)C(t)\,dt = x.
\]

Let us introduce the Lagrangian
\( \mathcal{L}(C, \lambda) = E \left[ \int_0^T \Gamma(s)U_t(C(s))ds \right] - \lambda E \left[ \int_0^T \zeta(s)C(s)ds - x \right] \) 

(9)

If \( C > 0 \), then for any \( \varepsilon = \varepsilon_s(\omega), \varepsilon_s(\omega) \in (0, C(s)) \), we have

\[
\left| \Gamma(s)U_t(C(s) + \varepsilon) - \lambda \zeta(s)(C(s) + \varepsilon) - \left[ \Gamma(s)U_t(C(s)) - \lambda \zeta(s)C(s) \right] \right| \\
\leq \left| \Gamma(s) \right| \frac{U_t(C(s) + \varepsilon) - U_t(C(s))}{\varepsilon} + |\lambda|\zeta(s) \leq \Gamma(s)U_t(\min \{ C(s), C(s) + \varepsilon \}) + |\lambda|\zeta(s) 
\]

(10)

the last inequality derives from the following two inequalities implied from the concavity of \( U_t \):

1. for \( \varepsilon > 0 \), we have \( U_t(C(s) - U_t(C(s) + \varepsilon) \geq U_t(C(s)) \cdot (\varepsilon) \), i.e., \( U_t(C(s) + \varepsilon) - U_t(C(s)) \leq U_t(C(s)) \cdot \varepsilon \);  
2. and for \( \varepsilon = \varepsilon_s(\omega) \), \( 0 < \varepsilon_s(\omega) < C(s) \), we have \( U_t(C(s) + \varepsilon) - U_t(C(s)) \geq U_t(C(s) + \varepsilon) \cdot \varepsilon \), i.e., \( U_t(C(s)) - U_t(C(s) + \varepsilon) \leq U_t(C(s) + \varepsilon) \cdot (\varepsilon). \)

If the right hand side of (10) has finite \( \lambda \times P \)-integral (\( \lambda \) here represents Lebesgue measure on \([0, T]\)), then the dominated convergence theorem ensures that \( \mathcal{L}(C, \lambda) \) is differentiable w.r.t. \( C \), and we can move the derivative into the sign of expectation and integral. For the same reason, \( \mathcal{L}(C, \lambda) \) has the similar conclusion w.r.t. \( \lambda \).

Using Lagrangian methods enable us to find out the point \( (C^{(1)}, \lambda^{(1)}) \) where the formal derivatives of \( \mathcal{L} \) w.r.t. \( (C, \lambda) \) are equal to zero. We can associate a unique portfolio \( \pi^{(1)} \) (up to equivalence) to \( C^{(1)} \) by Theorem 1.8 or Proposition 1.9 in [Jin, 2014], such that \( (\pi^{(1)}, C^{(1)}) \) is an admissible strategy. It remains to check carefully that this pair gives the maximum of \( J \).

From \( \Gamma(t)U_t(C(t)) = \lambda \zeta(t) \) (obtained by setting the formal derivative of \( \mathcal{L} \) w.r.t. \( C \) to equal to zero), and \( E \int_0^T \zeta(s)C(s)ds = x \), we get

\[
C^{(1)}(t) = I_1(\lambda^{(1)}\Delta(t)) 
\]

(11)

\[
E \int_0^T \zeta(s)I_1(\lambda^{(1)}\Delta(s))ds = x 
\]

(12)

Let’s introduce the function

\[
X_1(y) \triangleq E \int_0^T \zeta(s)I_1(y\Delta(s))ds = E \int_0^T \beta(s)I_1(y\Delta(s))ds, 0 < y < \infty 
\]

(13)

and assume that

\[
X_1(y) < \infty, y \in (0, \infty) 
\]

(14)

From now on, we should have the following additional assumption in force:

\( (A_1) \) For any \( \varepsilon > 0 \), \( \lambda \otimes P \{ Z(t) < \varepsilon \} > 0 \).

**Lemma 2.2.1**. Under the condition (2), \( X_1 \) inherit the convexity of \( I_1 \) on \((0, \infty)\);  

2. Under condition (14) and assumption \( (A_1) \), the function \( X_1(y) \) defined in (13) is continuous and strictly decreasing on \((0, \infty)\) with \( X_1(0) \triangleq \lim_{y \to 0^+} X_1(y) = \infty \),
$X_\gamma(\infty) \triangleq \lim_{y \to \infty} X_\gamma(y) = 0$.

**Proof.** Omitted. □

Therefore, $X_1$ has an inverse $Y_1 \triangleq X_1^{-1}$, and there is a unique $\lambda^{(1)} = Y_1(x)$ which satisfies (12) for any given $x > 0$. Then the corresponding consumption process in (6.7) becomes

$$C^{(1)}(t) = l_1(Y_1(x) \Delta(t)), \quad 0 \leq t \leq T$$

which belongs to $D(x)$, and according to Theorem 1.8 and Proposition 1.9 in [Jin, 2014], there is a unique (up to equivalence) portfolio process $\pi^{(1)}$ such that $\{\pi^{(1)}, C^{(1)}\} \in A(x)$, and the corresponding wealth process $X^{(1)}(t)$ is nonnegative on $[0, T)$ and vanishes at $t = T$, given by

$$X^{(1)}(t) = x - \int_0^t \beta(s)C^{(1)}(s)ds + \int_0^t \beta(s)(\pi^{(1)}(s))^T \sigma(s)d\tilde{W}(s)$$

$$+ \int_0^t \beta(s)(\pi^{(1)}(s))^T \rho(s)d\tilde{Q}(s)$$

If $U_1'(0) = \infty$, then $X^{(1)}(t)$ is positive on $[0, T)$. Indeed, at this time, $l_1(\infty) = 0$, and $l_1(\cdot) > 0$ on $[0, \infty)$. For $x \in (0, \infty)$, $0 < Y_1(x) < \infty$, and

$$\Delta(t) = Z(t)e^{\int_0^t (\gamma(u) - r(u))du} < \infty, \text{a.s. on } [0, T) \quad (\text{This is derived from } \infty < \infty < \infty, \text{a.s.})$$

since $EZ(t) = EZ(T) = 1$, $0 \leq t \leq T$, then $C^{(1)}(t) = l_1(Y_1(x) \Delta(t)) > 0$, a.s. on $[0, T)$.

Thus from the first equality of (16), we have

$$X^{(1)}(t) = \frac{1}{\beta(t)}\hat{E}\left[ \int_t^T C^{(1)}(s)\beta(s)ds \right] > 0, \quad \text{a.s., } t \in [0, T].$$

**Theorem 2.3** [Karatzas, 1989]. Assume that (14) holds. Then for any $x > 0$ and with $C^{(1)} \in D(x)$ given by (15), the pair $\{\pi^{(1)}, C^{(1)}\}$ constructed above belongs to $A(x)$ and is optimal for the problem of (7), i.e.,

$$V_1(x) = E\int_0^T \Gamma(s)U_1(C^{(1)}(s))ds$$

**Proof.** Omitted. □

In order to guarantee finiteness of the value function $V_1$ in (7) and to obtain a useful representation for it, we impose the condition

$$E\int_0^T \Gamma(s)|U_1(l_1(Y_1(\Delta(s))))|ds < \infty, \quad 0 < y < \infty$$

**Theorem 2.4.** Under the conditions (14) and (18), the value function $V_1(\cdot)$ satisfies

$$V_1(x) = G_1(Y_1(x)), \quad 0 < x < \infty$$

where the function $G_1 : (0, \infty) \to \mathbb{R}$ given by

$$G_1(y) \triangleq E\int_0^T \Gamma(t)U_1(l_1(y\Delta(t)))dt, \quad 0 < y < \infty$$

(20)
Furthermore, if $U_1$ also satisfies (2) and $l_1(\cdot)$ is continuous on $(0,\infty)$, then $X_1$ and $G_1$ are both continuously differentiable, as well as with

$$G_1'(y) = y X_1'(y), \quad 0 < y < \infty,$$  

(21)

and

$$X_1'(y) = \mathbb{E} \int_0^T \Gamma(t) \cdot \Delta^2(t) \cdot l_1'(y \Delta(t)) \, dt$$

(22)

be valid, and then

$$V_1'(x) = Y_1(x) \quad \text{and} \quad V_1''(x) = Y_1'(x), \quad 0 < x < \infty$$

(23)

follow from (19); in particular, $G_1$ is strictly decreasing, $V_1$ is strictly increasing, strictly concave and $V_1 \in C^2(0,\infty)$.

**Proof.** Omitted.

**Proposition 2.5.** If $U_1(0) > -\infty$, then (18) implies (14).

**Proof.** From the concavity of $U_1$, we have that

$$0 \leq C \cdot U_1(C) \leq U_1(C) - U_1(0) \leq |U_1(C)| + |U_1(0)|$$

for any $C \in [0,\infty)$, whence

$$y \Delta(t) l_1(y \Delta(t)) \leq |U_1(l_1(y \Delta(t)))| + |U_1(0)|$$

by taking $C = l_1(y \Delta(t))$ in the previous inequality. And from this inequality it follows that

$$y X_1'(y) = y \mathbb{E} \int_0^T \Gamma(t) \Delta(t) l_1(y \Delta(t)) \, dt$$

$$\leq |U_1(0)| \mathbb{E} \int_0^T \Gamma(t) \, dt + \mathbb{E} \int_0^T \Gamma(t) |U_1(l_1(y \Delta(t)))| \, dt < \infty, \quad 0 < y < \infty.$$  

Maximization of utility from terminal wealth

We now take up the problem of the maximization of the expected discounted utility from terminal wealth

$$J_2(x; \pi, C) \triangleq E[\Gamma(T) U_2(X(T))]$$

(24)

over the subclass

$$\mathcal{A}_2(x) \triangleq \{(\pi, C) \in \mathcal{A}(x) : E[\Gamma(T) U_2(X(T))] \leq \infty \}$$

(25)

and it is easy to verify that $\mathcal{A}_2(x) \equiv \mathcal{A}(x)$ if $U_2(0) > -\infty$. We denote the value function of the problem

$$V_2(x) \triangleq \sup_{(\pi, C) \in \mathcal{A}_2(x)} J_2(x; \pi, C)$$

(26)

This is in fact the complementary problem to that of §6. $U_2$ is the utility function.

Again, the case $x = 0$ is trivial. Indeed, for every $(\pi, C) \in \mathcal{A}(x) \equiv \mathcal{A}(0)$, we have $X(T) = 0$ a.s. from (1.31) in [Jin, 2014], and then $V_2(0) = U_2(0) \cdot \mathbb{E} \Gamma(T)$; this can be achieved by $\pi \equiv 0$ and $C \equiv 0$. So we only take $x > 0$ from now on.

In this setting, because utility comes now only from terminal wealth, the agent obviously tries to maximize the utility from his terminal wealth, within the constraints imposed by the level of his initial capital and quantified by the budget constraint, i.e.,

$$\mathbb{E}[X(T)\beta(T)] = \mathbb{E}[X(T)\zeta(T)] \leq x$$

(27)
which mandates that “the expected terminal wealth, discounted or deflated down to \( t = 0 \) (discounted or deflated down respectively in two different probability spaces), should not exceed the initial capital”.

Because utility comes now only from terminal wealth, it is quite reasonable that the latter should be increased within the constraints mandated by the level of the initial endowment as quantified by (27), by considering portfolio processes \( \pi \) in the class \( \mathcal{P}(x) \) of Definition 1.7 in [Jin, 2014].

**Proposition 3.1.** For every \( x > 0 \), we have
\[
V_2(x) = \sup_{\pi \in \mathcal{P}(x)} \mathbb{E}[\Gamma(T)J_2(x,T)].
\]

**Proof.** Omitted. \( \square \)

To solve the optimal terminal wealth problem

\[
V_2(x) = \sup_{\pi \in \mathcal{P}(x)} \mathbb{E}[\Gamma(T)U_2(x,T)]
\]

under the constraint
\[
\mathbb{E}[X(T)\beta(T)] = \mathbb{E}[X(T)\zeta(T)] = x,
\]

let us introduce the Lagrangian
\[
\mathcal{L}(x,\lambda) = \mathbb{E}[\Gamma(T)U_2(x,T)] - \lambda \mathbb{E}[X(T)\zeta(T) - x].
\]

Analogue to the previous section, let the formal partial derivatives of \( \mathcal{L} \) with respect to \( (X(T),\lambda) \) be equal to zero, we obtain

\[
\Gamma(T)U_2(x,T) = \zeta(T)\cdot \lambda,
\]

and then

\[
X(\Delta(T)) = l_z(\Delta(T)\lambda) \tag{28}
\]

where \( \lambda \) is decided by

\[
\mathbb{E}[l_z(\Delta(T)\lambda)] = x \tag{29}
\]

\( l_z \) is the inverse of \( U_2 \) (In order to make Lagrangian valid, we should have some supplementary conditions, cf. §6).

Let us now substantiate the heuristics of the preceding elementary Lagrangian multiplier considerations. We start by introducing the function

\[
J_2(y) = \mathbb{E}[\zeta(T)l_z(y\Delta(T))] = \mathbb{E}[\beta(T)l_z(y\Delta(T))], 0 < y < \infty \tag{30}
\]

and assume that for any \( y \in (0,\infty) \)

\[
J_2(y) < \infty \tag{31}
\]

And by the similar procedure discussed previously, we can prove that the solution of Lagrangian is the optimal, and it will make a maximum utility from terminal wealth.

**Maximization of utility from both consumption and terminal wealth**

We now consider an investor who derives utility both from “living well” (i.e., from consumption) and from “becoming rich” (i.e., from terminal wealth). His endowment is an initial positive wealth \( x \) but all the time he has to share his wealth according to a stock portfolio \( \pi(t) \) and a consumption rate \( C(t) \). His aim is to maximize the utility of his consumption and terminal wealth.

His expected total utility is then
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\[ J(x; \pi, C) \triangleq J_1(x; \pi, C) + J_2(x; \pi, C) = E \int_0^T \Gamma(t)U_1(C(t)) \, dt + E[\Gamma(T)U_2(X(T))] \]  (32)

and he tries to maximize \( J(x; \pi, C) \) over \( \mathcal{A}_{12}(x) \triangleq \mathcal{A}_1(x) \cap \mathcal{A}_2(x) \):

\[ V(x) \triangleq \sup_{(\pi, C) \in \mathcal{A}_{12}(x)} J(x; \pi, C) \]  (33)

Unlike the two problems studied already, this one calls for balancing competing objectives. Single-minded determination to maximize \( J_2(x; \pi, C) \) mandates no consumption at all. On the other hand, single-minded maximization of \( J_1(x; \pi, C) \) will leave the investor broke at the end (cf. Theorem 2.3 and Proposition 1.9 in [Jin, 2014]).

We will draw a proper compromise between these two competing objectives. At time \( t = 0 \) the investor simply divides the endowment \( x \) into two parts \( x_1 \geq 0 \) and \( x_2 \geq 0 \) with \( x_1 + x_2 = x \); for the amount \( x_1 \) (respectively, \( x_2 \)), the investor will face, form then on, an optimization problem with utility coming only from consumption (respectively, only from terminal wealth). It will be shown just how \( x_1, x_2 \) should be determined, in order for the resulting procedure to be optimal.

Throughout this section it will be assumed that \( U_1, U_2 \) are utility functions, with (2), (14), (18) and (31) hold, and Assumption (A1), (A2) also in force.

**Proposition 4.1.** For any \( x > 0 \) and an arbitrary portfolio/consumption pair \( (\pi, C) \in \mathcal{A}_{12}(x) \), let

\[ x_1 \triangleq E \int_0^T \beta(t)C(t) \, dt \]  (34)

Then there exists a pair \( (\hat{\pi}, \hat{C}) \in \mathcal{A}_{12}(x) \) such that

\[ J(x; \pi, C) \leq J(x; \hat{\pi}, \hat{C}) = V_1(x_1) + V_2(x - x_1) \]  (35)

In particular,

\[ V(x) \leq V_*(x) = \max_{x_1, x_2 \in (0, \infty)} V_1(x_1) + V_2(x_2) = \max_{y_1, y_2 \in (0, \infty)} G_1(y_1) + G_2(y_2) \]  (36)

**Proof.** Omitted.

**Corollary 4.2.** In the problem (33), the constraint condition can be an equality, i.e.,

\[ E\left[ \zeta(T)X(T) + \int_0^T \zeta(s)C(s) \, ds \right] = E\left( \beta(T)X(T) + \int_0^T \beta(s)C(s) \, ds \right) = x \]  (37)

**Proof.** Omitted.

Therefore, from Proposition 4.1, the question is to find \( x_1, x_2 \) for which the maximum in (36) is achieved, because then the total expected utility corresponding to the pair \( (\hat{\pi}, \hat{C}) \) in Proposition 4.1 will be exactly equal to \( V_*(x) \); this will in turn imply

\[ V(x) = V_*(x) \]  (38)

from (36), and thus \( (\hat{\pi}, \hat{C}) \) will be shown to be optimal for the problem of (33). In order to find out the above mentioned \( x_1, x_2 \), let

\[ \frac{d[V_1(x_1) + V_2(x_2)]}{dx_1} = 0, \]  i.e., \( \gamma_1(x_1) - \gamma_2(x_2) = 0 \), where \( x_2 = x - x_1 \). And because

\[ V_1'(x_1) - V_2'(x_2) = 0, \]  i.e., \( \gamma_1(x_1) - \gamma_2(x_2) = 0 \), where \( x_2 = x - x_1 \). And because
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\[ \frac{d^2}{dx_1^2} [V_1(x_1) + V_2(x_2)] = V_1''(x_1) + V_2''(x_2) = Y'_1(x_1) + Y'_2(x_2) < 0 \]

(i.e., \( V_1(x_1) + V_2(x_2) \) is strictly concave on \((0, \infty)\)), the maximum over \( x_1, x_2 \in (0, \infty), x_1 + x_2 = x \) indicated in (36) is obtained by \( x_1, x_2 > 0 \) which satisfy \( Y'_1(x_1) = Y'_2(x_2) \). And the constrained maximization in the last expression of (36) is achieved by

\[ y_1 = Y'_1(x_1) = Y'_2(x_2) = y_2 \triangleq y \]

where this common value is determined uniquely by

\[ X_1(y) + X_2(y) = x \]

(40)

(Uniqueness can be proved by the strict decrease of \( X_1, X_2 \). In other words, we find those values of \( x_1, x_2 \) for which the “marginal expected utilities” \( V_1(x_1), V_2(x_2) \) from the two individual optimization problems are identical.

In order to solve the problem (33) under the constraint of (37), we introduce the Lagrangian

\[ L(X, C, \lambda) = E \left[ \int_0^T \Gamma(t)U_1(C(t))dt + \Gamma(T)U_2(X(T)) - \lambda \left( \int_0^T \zeta(t)C(t)dt + \zeta(T)X(T) - x \right) \right]. \]

Imitate to the previous sections, let the formal derivatives of \( L \) with respect to \( (X(T), C, \lambda) \) be equal to zero, we obtain

\[ X^*(T) = l_2 \left( \Delta(T) \lambda^* \right), C^*(t) = l_1 \left( \Delta(t) \lambda^* \right) \]

(41)

where \( \lambda^* \) satisfies

\[ E \left[ \int_0^T \zeta(t)l_1 \left( \Delta(t) \lambda^* \right) dt + \zeta(T)l_2 \left( \Delta(t) \lambda^* \right) \right] = x \]

(42)

Compare with (40), \( \lambda^* \) in (42) is exactly the \( y \) in (40). The consequence of Lagrangian is identical with the preceding discussions (in order to make the Lagrangian feasible, we need some supplemental conditions, please refer to the corresponding part of previous discussion).

**Optimization problem in the case with deterministic coefficients**

The theory developed previously provides a precise characterization of the value function for the optimization problem, as well as explicit formulas for the optimal processes of consumption rate \( C^* \) and terminal wealth \( X^*(T) \). But for the optimal portfolio process \( \pi^* \), the “martingale methodology" that we have employed so far is able to ascertain only its existence; in general, there is no useful characterization that could lead to its computation. In this section supplementary assumptions of deterministic coefficients on the model enable us to obtain the optimal pair of portfolio/consumption in an explicit feedback form on the current level of wealth. Specifically, we shall assume throughout this section that \( \gamma(t), r(t), b(t), \sigma(t), \rho(t), \delta(t), \lambda(t) \) are all deterministic, and assume that \( U_1(0) > -\infty \), \( U_2(0) > -\infty \). And we will consider the consumption/investment problem with initial time \( t \geq 0 \). Furthermore, we shall assume that all stochastic processes in this section are \( \mathcal{F}_t \)-adapted (for the definition of \( \mathcal{F}_t \)).

We write the analogues of the wealth equation as
\[
dX(s) = (r(s)X(s) - C(s))ds + \pi^T(s)[b(s) + \delta(s) - r(s)]ds
+ \pi^T(s)x(s)dW(s) + \pi^T(s)v(s)dQ(s),
\]
\[X(t) = x > 0, \ 0 \leq t \leq s \leq T,\] (43)

and the value function
\[
V(t, x) = \sup_{(\pi, C) \in \mathcal{A}(t, x)} E \left[ \int_t^T \Gamma^t(s)U_1(C(s))ds + \Gamma^t(T)U_2(X(T)) \right],
\]
\[
V_1(t, x) = \sup_{(\pi, C) \in \mathcal{A}(t, x)} E \left[ \int_t^T \Gamma^t(s)U_1(C(s))ds \right],
\]
\[
V_2(t, x) = \sup_{(\pi, C) \in \mathcal{A}(t, x)} E \left[ \Gamma^t(T)U_2(X(T)) \right],
\]
where \( \mathcal{A}(t, x) \) consists of those admissible portfolio/consumption process pairs for which the corresponding wealth process \( X(s) \) decided by (43) remains nonnegative, a.s., and
\[
\Gamma^t(s) = e^{-\int_t^s \gamma(u)\,du} \quad \text{for} \quad 0 \leq t \leq s \leq T
\] (45)

Now we take the dynamic programming principle into account. For any \( \alpha \in [t, T] \), we have
\[
\int_t^T e^{-\int_t^s \gamma(u)\,du}U_1(C(s))\,ds + e^{-\int_t^T \gamma(u)\,du}U_2(X(T))
= \int_t^\alpha e^{-\int_t^s \gamma(u)\,du}U_1(C(s))\,ds + \int_\alpha^T e^{-\int_t^s \gamma(u)\,du}U_1(C(s))\,ds
+ e^{-\int_t^T \gamma(u)\,du}U_2(X(T))
= \int_t^\alpha e^{-\int_t^s \gamma(u)\,du}U_1(C(s))\,ds + e^{-\int_t^{\alpha} \gamma(u)\,du}\left[ \int_\alpha^T e^{-\int_t^s \gamma(u)\,du}U_1(C(s))\,ds
+ e^{-\int_t^T \gamma(u)\,du}U_2(X(T)) \right].
\]

Using dynamical programming principle, we have
\[
V(t, x) = \sup_{(\pi, C) \in \mathcal{A}(t, x)} E \left[ \int_t^\alpha \Gamma^t(s)U_1(C(s))\,ds
+ \Gamma^t(\alpha) \sup_{(\pi, C) \in \mathcal{A}(\alpha, X(\alpha))} E \left[ \int_\alpha^T \Gamma^\alpha(s)U_1(C(s))\,ds + \Gamma^\alpha(T)U_2(X(T)) \right] \right],
\]
(46)

where \( \Gamma^\alpha(\alpha)V(\alpha, X(\alpha)) \) is the discounted optimal value expected at time \( \alpha \) when we have used the control pair \( (\pi, C) \) and \( V(t, x) \) is the value function.

Thus for each pair \( (\pi, C) \in \mathcal{A}(t, x) \), (46) shows that
\[
V(t, x) \geq E \left[ \int_t^\alpha \Gamma^t(s)U_1(C(s))\,ds + \Gamma^t(\alpha)V(\alpha, X(\alpha)) \right] 
\] (47)

In this section, we should have the following assumptions in force.

\((A_3)\) \( V(s, y) \) defined in (44) is continuously differentiable with respect to \( s \).

Then by the previous discussion, \( V(s, y) \) is a function of twice-continuously differentiable in \( y \) and once-continuously differentiable in \( s \). Let us introduce some notation for later use.
\[ K(u, \pi, C, y) \triangleq U_1(C) + \frac{\partial V(u, y)}{\partial y} \left[ yr(u) - C + \pi^T (b(u) + \delta(u) - r(u)t) \right] + \frac{1}{2} \left( \frac{\partial^2 V(u, y)}{\partial y^2} \right) \left[ \pi^T \sigma(u) \right]^2 + \lambda(u) \left[ V(u, y + \pi^T \rho(u)) - V(u, y) - \pi^T \rho(u) \frac{\partial V(u, y)}{\partial y} \right], \] 

\( t \leq u \leq T, \pi \in \mathbb{R}^{d+1}, C \geq 0, y \geq 0, \)

\[ \text{LM}(u) \triangleq \int_t^u \Gamma^t(s) \frac{\partial V(s, X(s))}{\partial X} \pi^T(s) \sigma(s) dW(s) + \int_t^u \Gamma^t(s) \left[ V(s, X(s) - \pi^T(s) \rho(s)) - V(s, X(s) - \pi^T(s) \rho(s)) \right] dQ(s). \]

It is obvious that \( \text{LM}(u) \) is a local martingale. Apply Itô's formula to the process \( \Gamma^t(s)V(s, X(s)) \) on \( [t, t + \delta] \), \( 0 < \delta < T - t \), where \( X(s) \) is the solution of (43), we arrive to

**Theorem 5.1.** Suppose that \( \text{LM}(u) \), \( t \leq u \leq T \) is a martingale and \( (H_1) \) holds, then for any \( (t, x) \in [0, T] \times \mathbb{R}_+ \), \( V \) satisfies the following HJB (Hamilton-Jacobi-Bellman) equation

\[ \frac{\partial V(t, x)}{\partial t} - \gamma(t)V(t, x) + \sup_{(\pi, C) \in \mathcal{A}(t, x)} K(t, \pi(t), C(t), x) = 0, \quad 0 \leq t < T, \]

\[ V(T, x) = U_2(x), \]

and the optimal pair \( (\pi^*, C^*) \) can be found out by solving \( \sup_{(\pi, C) \in \mathcal{A}(t, x)} K(t, \pi(t), C(t), x) \) in (49).

**Proof.** Omitted. \( \square \)

**Corollary 5.2.** With all the conditions in Theorem 5.1, and the supplementary hypothesis that no jump occurs in the complete financial model, i.e., \( \bar{\sigma}(t) = \sigma(t) \) is a \( d \times d \) matrix, \( \rho(t) \equiv 0 \), and \( \sigma(t) \sigma^T(t) \) is strongly nondegenerate for any \( t \in [0, T] \), then the optimal pair \( (\pi^*, C^*) \) is given by

\[ C^*(t) = I(t)(Y(t, x)), \]

\[ \pi^*(t) = -[\sigma(t) \sigma^T(t)]^{-1} \frac{\partial H(t, x)}{\partial x} \left[ b(t) + \delta(t) - r(t)t \right]. \]

**Proof.** Omitted. \( \square \)

**CONCLUSION**

We get the formulae for optimal consumption and portfolio of investment in a financial market with jumps. And under some conditions, we can get the closed-form solutions for these problems although it is impossible in general. And we should emphasize that all the discussions were carried out under the condition that all stocks’ prices can be expressed by linear stochastic equations. However, the discussions can be extended to the situation when stocks’ prices are expressed by nonlinear stochastic equations. I will do this extension later.

**REFERENCES**


