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Equilibrium Analysis of an Advertising-Inventory-Pricing Competitive Model

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ABSTRACT

Using a market share attraction structure of advertising competition, this research introduces first, a parsimonious inventory-advertising model for which competition is limited to two symmetric firms and obtains its related comparative statics. Following a supermodular game approach, the derived comparative statics remain robust upon considering non-dominant firms competing in an asymmetric oligopoly of N rivals where each firm controls the inventory / advertising decision variables in addition to controlling price. The generalization appears applicable to both non-dominated oligopolies as well as dominated oligopolies. The study also derives closed form solutions for symmetric equilibrium advertising, price and order quantity together with related comparative statics.

KEYWORDS: Advertising, Economic order quantity, Pricing, Comparative statics, Market share attraction models; Supermodular games

INTRODUCTION

Researchers advocate building analytical models that are parsimonious, focusing on the truly important aspects of the problem to permit "what-if" analyses that allow both managers and policy makers to understand how changes in the parameters of their problem might affect their strategies or policies (viz. comparative statics / sensitivity analysis). Researchers also argue that such models should be *robust* in that their comparative statics predictions must hold up to a relaxation of the assumptions ([7], [20]).
Comparative statics, or the study of how the solutions of economic models change as the model parameters and specifications are changed, are important because (1) most of the testable predictions of economic theory are comparative statics predictions, (2) economic equilibrium analysis are often built from comparative statics analyses of the model's components, and (3) many of the insights and results needed by both managers and policy makers are obtained through comparative statics [3].
The functional areas of both marketing and operations represent the key value-adding processes of the modern business enterprise. It is these areas that are influential in specifying what should be produced, how it should be produced, and how to deliver said goods and services to the customers [18]. Thus, this paper begins by developing a model of inventory-advertising competition. As there is consensus that competition is an important phenomenon in today's markets, [24] said competition must be accommodated by using a market share attraction model of advertising competition. Holding the right amount of inventory under different circumstances is essential for achieving sound performance [7].
Whitin [28] was perhaps the first to jointly consider purchasing and marketing decisions by incorporating the effect of price-on-demand within the inventory model. Here the retailer must

decide the optimal price and order quantity. The price-dependent demand model was followed by another stream of research to integrate the inventory model with advertising-dependent demand ([9]; [13]; [25]; [27]). While the aforementioned research considered only a single firm, there have been a few studies within a static framework on advertising competition in duopoly / oligopolies ([10]; [15]; [16]; [21]). The above studies, however, do not consider inventory related costs in the modeling effort.

The present study has four main objectives that have not yet been fully addressed. The first objective is to investigate analytically how a firm would adapt optimum order quantity and optimum advertising expenditure in response to changes in any of its own parameters, or competitors' parameters in a symmetric duopoly market. Second, in a truly asymmetric non-dominated oligopolistic market for which every market share is less than 50%, should competitors adjust their order quantities and advertising expenditures in response to changes in market conditions in a manner consistent, or inconsistent, with a symmetric duopoly for which the two competitors share the market equally? The third objective is to compare the normative behaviors of rivals in non-dominated oligopolies and their counterparts related to dominated oligopolies (where one firm has a market share equal to or larger than 50%) in response to changes in model parameters upon the inclusion of the additional variable of price into the analysis. The final objective questions the literature (e.g., [12]) which states that the defensive strategy of non-dominated firms will be to decrease the equilibrium price and quantity of advertising in response to the competitive entry in an oligopoly of fixed market potential. This paper asks, do the results continue to be robust when including in the model the operations variable of order quantity?

The next section of the paper outlines a theoretical model that incorporates advertising and order quantity. The third section derives the comparative statics for a symmetric duopoly. The fourth section reports the results of applying a supermodular games approach to self-comparative statics as a tool for assessing the signs of comparative statics derived for non-dominated asymmetric oligopolies. The fifth section compares the normative behaviors of rivals in response to changes in model parameters for two alternative oligopoly structures in the presence of price. The penultimate section addresses the strategy implications regarding the incumbents' response to new entry. The final section summarizes and concludes the paper. Derivation of key formulas and mathematical proofs of five propositions are reported in an Appendix available upon request.

A MATHEMATICAL MODEL OF INVENTORY-ADVERTISING COMPETITION

This research considers simple one-step competing supply chains in which each supplier (vendor) strives to meet an advertising-dependent demand of potential customers. In doing so, the inventory replenishment process of each supplier of a differentiated product is represented by an economic order quantity (EOQ) model for which its classical assumptions are satisfied (mainly, the demand rate is deterministic and constant, shortages are not allowed, and the order quantity is the same for each order). It is further assumed that any intermediate levels in each chain coordinate their activities fully with their related supplier. Each firm aims at maximizing its profit per unit of time non-cooperatively with other firms.

The model described both in this section and in the one to follow, considers first (as in Little [14] and Mesak [15]) a market share attraction model in a duopoly market, with one advertising spending variable x_j measured in real dollars per unit time for each firm j, where j = 1, 2. Sales of firm 1, $D_1(x_1, x_2)$, in a market of two competitors is expressed as follows:

$$D_1(x_1, x_2) = \frac{m f(x_1)}{f(x_1) + f(x_2)} = \frac{m \beta_1 x_1^{\delta_1}}{\beta_1 x_1^{\delta_1} + \beta_2 x_2^{\delta_2}} \text{ , and } D_2(x_1, x_2) = m - D_1(x_1, x_2), \quad (1)$$

where m is the market potential (considered constant for a mature market) and $f(x_1)$ and $f(x_2)$ are the attraction functions of firms 1 and 2, respectively. The theoretical justification of the advertising-driven market share function associated with $D_j(x_1, x_2)$ given by (1) is discussed in [5]. While some other elaborate forms of $f(x_j)$ can be used, their implications appear to be empirically consistent (see [17]). All rivals are assumed to have concave advertising attraction functions, such that $f'(x_j) > 0$ and $f''(x_j) < 0$ for $x_j > 0$. For all the attraction functions, the measures of advertising effectiveness β's and the shaping parameters δ's are strictly positive parameters, and for a concave attraction function $0 < δ < 1$. The terms are defined as follows:

P_j = Selling price per unit of firm j
MC_j = Marginal cost per unit of firm j
y_j = Profit margin per unit sold of firm j: $(P_j - MC_j)$
$C(x_j)$ = Advertising cost function of firm j
Q_j = Order quantity of firm j
C_{oj} = Ordering cost per order of firm j
C_{hj} = Inventory holding cost per unit held per unit time of firm j
F_j = Fixed cost of firm j
m = Market potential (considered constant for a mature market),

Consistent with most empirical industrial organization studies, each firm strives to maximize profit. Based on the above definition of terms, the profit function π_j of firm j per unit time is given by

$$\pi_j = \gamma_j D_j(x_1, x_2) - C(x_j) - \frac{Q_j}{2} C_{hj} - \frac{D_j(x_1, x_2)}{Q_j} C_{oj} - F_j, \quad j = 1, 2. \tag{2}$$

It is also assumed that the advertising cost function is convex and of the form $C(x_j) = d_j x_j^{\varepsilon_j}$, where d_j and ε_j are positive constants such that $d_j > 0$, and $\varepsilon_j \geq 1$ [1] with underlying properties, $C'(x_j) > 0$ and $C''(x_j) \geq 0$, for $x_j > 0$. Defining $\Pi_j = \pi_j / \gamma_j m$, for convenience of analysis, and substituting for $D_j(x_1, x_2)$, j = 1, 2, expression (2) takes the following form:

$$\Pi_1 = \frac{f(x_1)}{f(x_1) + f(x_2)} \left[1 - \frac{C_{o1}}{\gamma_1 Q_1} \right] - \frac{C(x_1)}{\gamma_1 m} - \frac{C_{h1}}{2\gamma_1 m} Q_1 - \frac{F_1}{\gamma_1 m}. \tag{3}$$

$$\Pi_2 = \frac{f(x_2)}{f(x_1) + f(x_2)} \left[1 - \frac{C_{o2}}{\gamma_2 Q_2} \right] - \frac{C(x_2)}{\gamma_2 m} - \frac{C_{h2}}{2\gamma_2 m} Q_2 - \frac{F_2}{\gamma_2 m}. \tag{4}$$

The analysis of this situation is based on the Nash equilibrium solution concept of game theory. Researchers have widely used the Nash equilibrium solution concept in an oligopoly (e.g., [11]; [15]; [21]). The Nash equilibrium solution requires that no single rival unilaterally changes its advertising budget and order quantity given the other rivals' optimal advertising budgets and order quantities. Information on each firm's demand function, cost structure, and decision rules is common knowledge to all firms. The first-order optimality conditions for an interior solution ($x_j^* > 0$ and $Q_j^* > 0$) implies that

$$\frac{\partial \Pi_j}{\partial x_j} = 0, \text{ and } \frac{\partial \Pi_j}{\partial Q_j} = 0, \quad j = 1, 2. \tag{5}$$

Also to ensure that expressions (5) indicate profit maximization, both second-order derivatives $\partial^2 \Pi_j / \partial x_j^2$ and $\partial^2 \Pi_j / \partial Q_j^2$ must be negative, and $\frac{\partial^2 \Pi_j}{\partial x_j^2} \frac{\partial^2 \Pi_j}{\partial Q_j^2} - \left(\frac{\partial^2 \Pi_j}{\partial x_j \partial Q_j} \right)^2$ must be positive for

all *j* (a complete derivation of the first and second order conditions of optimality for firms 1 and 2 are presented in a separate Appendix available upon request). Comparative statics are derived next for a symmetric duopoly.

COMPARATIVE STATICS FOR A SYMMETRIC DUOPOLY

This section provides a theoretical derivation of the impact of changes in each of the shift parameters, namely γ_j, β_j, δ_j, m, d_j, ϵ_j, C_{hj}, C_{oj} for a given rival *j* on its optimal advertising expenditure x_j^* and optimal order quantity Q_j^* (self-comparative statics), as well as on similar quantities related to the competitors (cross – comparative statics). In a symmetric duopoly, firms (which are highly similar in all economic respects) produce a storable, homogeneous product. These firms compete against each other for the same buyers. Symmetric competition also stipulates that rivals will have the same production and inventory costs, acquire real promotion on the same terms, charge the same fixed price, and face symmetric demand functions [32]. Therefore, all comparable shift parameters are assumed to be equal and firms are assumed to have equal market shares.

Although equations (5) for a symmetric competitive structure could be solved explicitly for x_j^* and Q_j^* in terms of model parameters, several of the self-comparative statics obtained afterwards would have been missing and not all of the cross-comparative statics would have been disclosed. Therefore, the implicit function theory (IFT) (see Bertsekas [6]) is instead used to arrive at expressions for self and cross comparative statics. Because of the curse of dimensionality when the number of decision variables per firm is more than one and when the number of competitors $N \geq 3$, the theoretical analysis is confined, for now, to a duopoly. The generality of the obtained results to a non-dominated asymmetric oligopoly is assessed in the next section.

To study the sensitivity of each x_j^* and Q_j^* (where *j* = 1, 2), to a change in one of the model parameters Θ, equations (5) are partially differentiated with respect to Θ and then equated to zero to obtain:

$$\frac{\partial^2 \Pi_1}{\partial x_1^2}\frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_1}\frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial x_2}\frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_2}\frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} = 0 \ ,$$

$$\frac{\partial^2 \Pi_1}{\partial Q_1 \partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1^2}\frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1 \partial x_2}\frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1 \partial Q_2}\frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1 \partial \theta} = 0 \ ,$$

$$\frac{\partial^2 \Pi_2}{\partial x_2 \partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial x_2 \partial Q_1}\frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial x_2^2}\frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial x_2 \partial Q_2}\frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial x_2 \partial \theta} = 0 \ , \text{ and}$$

$$\frac{\partial^2 \Pi_2}{\partial Q_2 \partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2 \partial Q_1}\frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2 \partial x_2}\frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2^2}\frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2 \partial \theta} = 0 \ . \tag{6}$$

Here, it is also assumed that all second partial derivatives of Π_j with respect to each of the equilibrium values x_j^* and Q_j^* together with each of the *shift* parameters Θ do exist and are continuous. The system of equations (6) can be put in the following matrix form:

$$
\begin{bmatrix}
\dfrac{\partial^2 \Pi_1}{\partial x_1^2} & \dfrac{\partial^2 \Pi_1}{\partial x_1 \partial Q_1} & \dfrac{\partial^2 \Pi_1}{\partial x_1 \partial x_2} & \dfrac{\partial^2 \Pi_1}{\partial x_1 \partial Q_2} \\[2mm]
\dfrac{\partial^2 \Pi_1}{\partial Q_1 \partial x_1} & \dfrac{\partial^2 \Pi_1}{\partial Q_1^2} & \dfrac{\partial^2 \Pi_1}{\partial Q_1 \partial x_2} & \dfrac{\partial^2 \Pi_1}{\partial Q_1 \partial Q_2} \\[2mm]
\dfrac{\partial^2 \Pi_2}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \Pi_2}{\partial x_2 \partial Q_1} & \dfrac{\partial^2 \Pi_2}{\partial x_2^2} & \dfrac{\partial^2 \Pi_2}{\partial x_2 \partial Q_2} \\[2mm]
\dfrac{\partial^2 \Pi_2}{\partial Q_2 \partial x_1} & \dfrac{\partial^2 \Pi_2}{\partial Q_2 \partial Q_1} & \dfrac{\partial^2 \Pi_2}{\partial Q_2 \partial x_2} & \dfrac{\partial^2 \Pi_2}{\partial Q_2^2}
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial x_1}{\partial \theta} \\[2mm]
\dfrac{\partial Q_1}{\partial \theta} \\[2mm]
\dfrac{\partial x_2}{\partial \theta} \\[2mm]
\dfrac{\partial Q_2}{\partial \theta}
\end{bmatrix}
= -
\begin{bmatrix}
\dfrac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} \\[2mm]
\dfrac{\partial^2 \Pi_1}{\partial Q_1 \partial \theta} \\[2mm]
\dfrac{\partial^2 \Pi_2}{\partial x_2 \partial \theta} \\[2mm]
\dfrac{\partial^2 \Pi_2}{\partial Q_2 \partial \theta}
\end{bmatrix}.
\tag{7}
$$

Designating the 4 x 4 square matrix of the second-order partial derivatives in (7) by matrix **H**, equations (7) can be rewritten as:

$$
\begin{bmatrix}
\dfrac{\partial x_1}{\partial \theta} \\[2mm]
\dfrac{\partial Q_1}{\partial \theta} \\[2mm]
\dfrac{\partial x_2}{\partial \theta} \\[2mm]
\dfrac{\partial Q_2}{\partial \theta}
\end{bmatrix}
= - \, \boldsymbol{H^{-1}}
\begin{bmatrix}
\dfrac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} \\[2mm]
\dfrac{\partial^2 \Pi_1}{\partial Q_1 \partial \theta} \\[2mm]
\dfrac{\partial^2 \Pi_2}{\partial x_2 \partial \theta} \\[2mm]
\dfrac{\partial^2 \Pi_2}{\partial Q_2 \partial \theta}
\end{bmatrix}
\tag{8}
$$

The derivations of the elements of matrix **H** and all other terms related to expressions (8) are shown in a separate Appendix. The Appendix also shows that matrix **H** is nonsingular, so its inverse does exist. Gruca et al. [12] assert that negative semi-definiteness of the square matrix **H** is essential in ensuring uniqueness of the Nash equilibrium solution of the system of equations (5). A proof of the negative semi-definiteness of matrix **H** is provided in the Appendix. Based on the solution of equations (8) for a symmetric duopoly, the following proposition is introduced with related terms defined below [Proof is found in a separate Appendix].

$$
a = \frac{f'}{4f} \frac{C_o}{\gamma Q^2} \quad \text{(positive term)},
\tag{9}
$$

$$
b = \frac{\left(f'' f - f'^2\right)}{4 f^2}\left[1 - \frac{C_o}{\gamma Q}\right] - \frac{C''}{\gamma m} \quad \text{(negative term), and}
\tag{10}
$$

$$
\Delta_1 = -\frac{b C_o}{\gamma Q^3} - a^2 \quad \text{(positive term)}.
\tag{11}
$$

Proposition 1: *For a symmetric duopoly*

(i)   *An increase in the parameters γ, β, or δ of a firm should elicit an increase in the equilibrium advertising and order quantity internally, but cause a decrease in the equilibrium advertising and order quantity of its competitor.*

(ii)   *An increase in either of the parameters d or ϵ of a firm should elicit a decrease in the equilibrium advertising and order quantity internally, but cause an increase in the equilibrium advertising and order quantity of its competitor.*

(iii)   *An increase in parameter C_h of a firm should elicit a decrease in the equilibrium advertising and order quantity internally, but cause an increase in the equilibrium advertising and order quantity of its competitor.*

(iv)   *An increase in parameter C_o of a firm should elicit a decrease in the equilibrium advertising and an increase (decrease) in the equilibrium order quantity if*
$\Delta_1 - 2a^2 > (<)0$ *internally, but cause an increase in the equilibrium advertising and order quantity of its competitor.*

Part (i) of the above proposition dictates that a firm should favor increasing both its advertising spending and order quantity to enhance profits whereas its competitor would be better off by doing the exact opposite when the above parameters increase. Part (ii) implies that if the parameters associated with the advertising cost function increase, an optimal response for the firm is to reduce its advertising spending and order quantity thus mitigating the impact. However, the competitor should consider this situation as an opportunity to increase its advertising expenditure and order quantity in order to increase profit.
Part (iii) indicates that if the inventory holding cost rate per unit increases, the firm's optimal response should be to decrease its advertising expenditure and order quantity. The firm's competitor, in response, would increase its advertising expenditure and order quantity. If the ordering cost per order of a firm increases, as described in part (iv), the firm should respond by decreasing the advertising expenditure; however the optimal response in the change of order quantity will be dictated by other parameters. The competitor's optimal policy will be to take advantage of the situation by increasing both the advertising expenditure and order quantity. Note that the signs of the comparative statics relating the changes in the marketing parameters (γ, β, δ, *d, and* ϵ) to changes in optimal advertising are similar to those reported in Mesak [24], although that study does not consider inventory related costs. The remaining comparative statics are unique to the present study. In particular, changes in the above marketing parameters are found to affect the operations decision variable, Q^*. Furthermore, changes in the operations parameters (C_0 and C_h) are found to affect the marketing decision variable, x^*. It is conjectured here that the results depicted in the proposition are generalized to a non-dominated asymmetric oligopoly of N firms for which each firm has a market share less than 50%.
The theoretical findings reported in the above proposition are derived under ideal symmetric conditions that rarely, if ever, materialize in practice. In the next two sections (based in part on Sections 2 and 4 of Mesak et al.[18]), the article will assess the robustness of the theoretical sensitivity results to deviations from ideal conditions through examining the applicability of these results to a practical non-dominated oligopoly setting.

SUPERMODULAR GAMES APPROACH TO SELF-COMPARATVE STATICS

The proposition reported in this section describes the impact of changes in each of the shift parameters (namely γ_j, β_j, δ_j, d_j, ϵ_j, C_{hj}, C_{oj}) for a given firm j on its optimal advertising expenditure x_j^* and optimal order quantity Q_j^* (self-comparative statics) in a non-dominated, asymmetric oligopoly for which the market share of each firm is less than 50%.
Ease of use and flexibility in solving for comparative statics in game-theory problems have made monotone methods popular in the economics literature ([2]; [26]). However, they are still (for all intents and purposes) absent from marketing-operations interface related applications. Therefore, the definition of a supermodular game and its related comparative statics implications are briefly discussed below.
Definition: N firms compete in a smooth supermodular game if the following conditions hold for each firm j and each competitor i:
(C.1) *Differentiability*: The profit equation is twice continuously differentiable with respect to x_j and Q_j.
(C.2) *Complementary Strategies* (of each firm's own strategic variables): $\partial^2\pi_j/\partial x_j\partial Q_j \geq 0$.

(C.3) *Strategic Complements* (for strategic variables between firms):
$$\partial^2\pi_j/\partial x_j\partial x_i \geq 0, \; \partial^2\pi_j/\partial x_j\partial Q_i \geq 0, \; \partial^2\pi_j/\partial Q_j\partial x_i \geq 0 \text{ and } \partial^2\pi_j/\partial Q_j\partial Q_i \geq 0. \qquad (12)$$
(C.4) *Complementary Policy Parameter θ*:
$$\partial^2\pi_j/\partial x_j\partial\theta \geq 0 \text{ and } \partial^2\pi_j/\partial Q_j\partial\theta \geq 0. \qquad (13)$$
One can think of this as a game of super-complementarity, which implies that there is complementarity among all strategies and a policy parameter. The important thing to notice is that this specification implies increasing marginal returns between all possible pairs of choice variables and the policy parameter. Under these conditions, and assuming a unique Nash equilibrium, the following comparative statics results hold for all firms (Milgrom and Roberts, Theorem 6 and its Corollary [29]):
$$\partial x_j^*/\partial\theta \geq 0; \; \partial Q_j^*/\partial\theta \geq 0. \qquad (14)$$
That is, when conditions (C.1) – (C.4) are met, an increase in parameter θ will have a non-negative effect on the Nash advertising budgets and size of the order quantity. It should also be mentioned that when *all* the inequalities in (C.2) – (C.4) are in the opposite direction, the inequalities in (14) are reversed to '≤ 0' (Amir [2]).
Upon employing a supermodular games approach to sensitivity analysis, the obtained self - comparative statics are summarized in Proposition 2 (proven in a separate Appendix).

Proposition 2. *For a non-dominated asymmetric oligopoly*

(i)   *An increase in either of the firm's parameters γ, or β should elicit no decrease in the equilibrium advertising nor in the order quantity. Furthermore, an increase in the parameter δ of a firm should elicit no decrease in x^δ nor in the equilibrium order quantity Q.*

(ii)   *An increase in either of the parameters d or ε of a firm should elicit no increase in either the equilibrium advertising or the order quantity.*

(iii)   *An increase in parameter C_h of a firm should elicit no increase in either the equilibrium advertising or the order quantity.*

(iv)   *An increase in parameter C_o of a firm should elicit no increase in equilibrium advertising and elicit no decrease in the equilibrium order quantity.*

The self-comparative statics reported in Proposition 2 for a non-dominated asymmetric oligopoly (market share of each firm is less than 50%) is consistent with those derived for a symmetric duopoly reported in Proposition 1.

The next section attempts to assess the robustness of the theoretical sensitivity results pertaining to optimal advertising and order quantity to a competitive setting that incorporates the additional variable of price. For the related model, the attraction of each firm depends on both advertising x and price P. Self-comparative statics are only derived for two distinct oligopoly structures using a supermodular game approach. Mesak et al. [18] show that in terms of magnitude, a change in a parameter of a rival has a much more significant impact on the equilibrium measures of its own than on those of its competitors.

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In section 2 the attraction of firm j, A_j, can be put in the form $A_j = f(x_j) = Exp\,(a_j + \delta_j\,\ln x_j)$, where $\beta_j = Exp(a_j)$. For a multinomial logit market share model (MNL) that also includes price P_j, the attraction function takes on the form $A_j = Exp\,(a_j + \delta_j\,\ln x_j - c_j\,P_j)$, $a_j > 0$, $\delta_j > 0$, and $c_j > 0$ (Basuroy and Nguyen [4]), so that expression (2) takes on the following form:

$$\pi_j = m (P_j - MC_j - C_{oj} / Q_j) \, Exp \, (a_j + \delta_j \ln x_j - c_j P_j) / (A_j + \Sigma_{i \neq j} A_i) - d_j x_j^{ej} - Q_j C_{hj} / 2 - F_j. \qquad (15)$$

The two propositions reported in this section describe the impact of changes in each of; i) the operations parameters (MC_j, C_{oj}, C_{hj}), ii) the advertising cost parameters (d_j, ϵ_j) and iii) the attraction function parameters (a_j, δ_j , c_j) for a given rival j on it's a) optimal advertising expenditure x_j^*, b) optimal price P_j^* and c) optimal order quantity Q_j^* (self-comparative statics) in a non-dominated oligopoly for which the market of each firm is less than 50%, and a dominated asymmetric oligopoly for which one firm has a market share larger than or equals to 50%. Because of the curse of dimensionality (dealing with more than two decision variables per firm and the difficulty in meeting all related conditions when applying a supermodular game approach for comparative statics) and exclusively for Proposition 3 associated with non-dominant rivals, comparative statics are obtained for symmetric firms.

Proposition 3. *For a non-dominated symmetric oligopoly*
  i.   *An increase in either of the parameters c, d, or ε of a firm should elicit no increase in internal equilibrium advertising.*

  ii.   *An increase in the parameter δ of a firm should elicit no decrease in internal equilibrium advertising.*

  iii.   *An increase in either of the parameters MC, C_0, or C_h of a firm should elicit no decrease in the internal equilibrium price.*

  iv.   *An increase in the parameter c of a firm should elicit no increase in the internal equilibrium price.*

  v.   *An increase in the parameter C_h of a firm should elicit no increase in the internal equilibrium order quantity.*

  vi.   *An increase in the parameter C_0 of a firm should elicit no decrease in the internal equilibrium order quantity.*

Having demonstrated that rivals of a non-dominated oligopoly should adjust their equilibrium advertising and order quantity in response to changes in the parameters *MC*, (or - *γ*), *δ, d, ϵ, C_o* and *C_h* in a manner consistent with Proposition 1, one may ask: Are the behaviors of rivals in a dominated oligopoly similar to those of a non-dominated oligopoly? The answers to the above question are documented in Proposition 4.

Proposition 4. *For a dominated asymmetric oligopoly*
  i.   *An increase in either of the parameters MC, C_o, C_h, d, or ϵ of the dominant firm should elicit responses in a manner consistent with the findings of Proposition 3.*

  ii.   *An increase in the parameter c of the dominant firm should elicit no internal decrease in equilibrium advertising, no internal increase in equilibrium price, nor an internal increase in order quantity.*

  iii.   *An increase in either of the parameters a, or δ of the dominant firm should elicit no internal increase in equilibrium advertising, but elicit no internal decrease in either equilibrium price, or order quantity.*

Propositions 1 through 4 taken together imply that the self-comparative statics related to equilibrium advertising and order quantity for a symmetric duopoly, in the context of the MNL market share model, are generalized to non-dominant firms even upon the incorporation of the additional price variable in the model. However, for a dominated oligopoly, the normative behavior of the dominant firm appears different from the normative behaviors of its non-dominant rivals when it comes to the marketing parameters δ and c associated with the attraction function as well as the operations parameter C_h associated with inventory holding costs as summarized in Table 1.

The next section examines comparative statics related to entry for a symmetric oligopoly in which the marketing variables of advertising and price together with the operations variable of order quantity are all considered.

Table 1: Comparative Statics for Non-dominant and Dominant Firms

	Non-dominant Firm			Dominant Firm		
	x_j^*	P_j^*	Q_j^*	x_j^*	P_j^*	Q_j^*
MC_j	≤ 0	≥ 0	N.O.	≤ 0	≥ 0	≥ 0
C_{oj}	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0	≥ 0
C_{hj}	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
d_j	≤ 0	N.O.	N.O.	≤ 0	≥ 0	≥ 0
ϵ_j	≤ 0	N.O.	≥ 0	≤ 0	≥ 0	≥ 0
a_j	N.O.	N.O.	N.O.	≤ 0	≥ 0	≥ 0
δ_j	≥ 0	N.O.	N.O.	≤ 0	≥ 0	≥ 0
c_j	≤ 0	≤ 0	N.O.	≥ 0	≤ 0	≤ 0

N.O. stands for "Not Obtained"

COMPARATIVE STATICS RELATED TO ENTRY

In this section, closed form solutions for symmetric equilibrium advertising, price and order quantity are obtained together with related comparative statics with respect to competitive entry. The first order conditions for optimality regarding (15), $\partial \pi_j / \partial x_j = \partial \pi_j / \partial P_j = \partial \pi_j / \partial Q_j = 0$ imply for a symmetric oligopoly that:

$$x^* = (\delta m / N c d \epsilon)^{1/\epsilon}, \quad P^* = N/c(N-1) + MC + (N C_o C_h / 2m)^{1/2}, \text{ and } Q^* = (2m C_o / N C_h)^{1/2} \tag{16}$$

For (16) to represent a unique symmetric Nash equilibrium, it is shown in the Appendix that the Heissian matrix $H_{3 \times 3}$ of second order partial derivatives of π with respect to x, P and Q is negative semi-definite. Substituting for x^*, P^* and Q^* from (16) into (15) and defining optimal oligopoly profit $\pi_o^* = N \pi^*$ produces

$$\pi_o^* = m N/c(N-1) - (\delta m / c \epsilon) - (m N C_o C_h / 2)^{1/2} - NF. \tag{17}$$

Based on (16) and (17), we are in a position to introduce Proposition 5.

Proposition 5. *For a symmetric oligopoly and in the context of the MNL market share model, an increase in the number of competitors N:*
   i.   *Decreases the equilibrium measures x*, Q* and π_o*.*

   ii.   *May increase the equilibrium measure P*.*

When it comes to the marketing variable of price, the related result in Proposition 5 (ii) is inconsistent with those reported in Gruca et al. [12], and Basuroy and Nguyen [4]. The above studies consider non-dominated brands in the context of market share attraction models. Further, those studies illustrates falling advertising and price in response to competitive entry. However, they do not incorporate any operations variables in the modeling effort. Proposition 5 (ii) has empirical support. Frank and Salever [8] show that incumbent producers of prescription drugs raise the price after generic entry, while generic incumbents reduce the price after generic entry. There is also support on theoretical grounds for the content of Proposition 5 (ii) based on a symmetric inventory-price model articulated by Min and Chen [19]. The above authors, however, consider C_o and C_h as decision variables in addition to price, and do not incorporate advertising in their competitive model.

CONCLUSIONS, IMPLICATIONS AND DIRECTIONS FOR FUTURE RESEARCH

In short, this research presents a parsimonious inventory-advertising model for which competition is limited to two symmetric firms and derives its comparative statics predictions (Proposition 1). Using a "supermodular games approach", considered a novel application in the operations-marketing interface literature, the above comparative statics are generalized to a non-dominated asymmetric oligopoly of N competing firms, each possessing market shares of less than 50% (Proposition 2). Further generalization for non-dominant firms is materialized when the additional marketing variable of price is incorporated into the attraction function of competing firms (Proposition 3). However, for a dominated oligopoly, the normative behavior of the dominant firm appears different from the normative behaviors of its non-dominant rivals when it comes to changes in the parameters of the attraction function and inventory cost (Proposition 4 and Table 1). The defensive reaction of symmetric incumbents to competitive entry are obtained in Proposition 5. To that end, the main objectives of this study have been achieved.
The main conclusion drawn from the findings is of significant strategic appeal to marketing and production / purchasing managers. Within the context of an MNL market share model, for an oligopolistic market that is not dominated by a single rival (largest market share is less than 50 %), a firm would adjust its advertising spending, order quantity, and price in response to changes in its own shift parameters in a manner consistent with the comparative static properties reported in Propositions 1 and 3. Relaxing the assumption of symmetry would only complicate the analysis without yielding additional insight. However, for a firm that dominates its industry, different courses of action are required for changes related to the parameters of the attraction function and of holding inventory cost. Finally, for non-dominant firms the defensive strategy of decreasing the equilibrium advertising and price in response to competitive entry for a fixed market potential do not necessarily remain implementable upon the inclusion of the operations variable of order quantity in the modeling effort. The studied model offers a framework for further research. Future research would investigate the linkage between our findings and empirical firms' behavior within several industries. In this work we have assumed aggregation of advertising over all media. The current study utilizes a basic EOQ model within a

profit maximizing framework, which can later be extended to include taxes and inflation within the model. An interesting area for future research would be to consider different advertising media of various effectiveness and to examine synergy effects among them ([22]). Following the study of Sivakumar [24] for asymmetric competition, the present study can be extended to derive normative defensive strategies in growing markets related to an oligopoly for which one firm dominates the market (its market share is equal to or greater than 50%) while all rivals share the rest of the market equally. In this regard, the dominant firm would have at least one parameter different from the non-dominant competitors. Comparable parameters of non-dominant competitors, however, would be considered to be similar.

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