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A Note on Modeling Product Allocation between Brick-and-Mortar and Online Stores

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ABSTRACT

A mathematical programming model is developed and analytically solved in this study for the problem of optimally allocating a product by a monopolistic retailer to its brick-and-mortar and online stores under uncertain aggregate demand following a uniform probability distribution. Impacts of changes in six key model parameters on retailer's profitability are analytically examined.

KEYWORDS: Product allocation, Optimization, Simulation

INTRODUCTION

There has been a dramatic increase in the volume and coverage of goods and services sold online over the last two decades. While e-commerce was virtually nonexistent in the early 1990s, online sales stood at \$263.3 billion and accounted for 5.6 percent of total retail sales in the U.S. economy in 2013 (Gorodnichenko et al., 2015). Major retailers such as Wal-Mart and Target currently operate not only brick-and-mortar stores but also online stores. In fact, online stores offer the convenience of consumer purchasing and allow retailers to reach those customers whom they were previously less likely to attract.

Some distinctive features of the offline and online market segments are noted in the literature. For example, Chu et al. (2007) find that across 12 vastly different product categories, consumers exhibit lower price sensitivities when they shop online than when they shop offline. Gorodnichenko et al. (2015) report that online prices are more flexible than prices charged in conventional stores. Fassnacht and Unterhuber (2016) provide experimental evidence that price differentiation with lower online prices is feasible. Because of the differences, a profit-maximizing retailer needs to resolve the strategic issue of optimally allocating a product for sale to its brick-and-mortar and online stores.

Our paper intends to tackle a problem of product allocation in a market, where a sole retailer, having achieved a monopolistic position through differentiation (Lee and Ng, 2001), sets prices in the offline and online segments and then allocates a certain quantity of a product to each segment to improve its profitability. Three notable studies are relevant to this paper. Lee and

Ng (2001) addressed the issue of optimal service capacity allocation and derived the optimal scheme of capacity allocation in a monopolistic market comprised of two segments under deterministic demand. Deng et al. (2008) studied capacity allocation in a multi-segment monopolistic market under uncertain demand following a Poisson distribution. They proposed the optimal policies of allocating capacity to the segments of higher revenue. Zeng (2013) developed multiple pricing strategies for a retailer who operates in a market composed of experienced and inexperienced consumers. Our paper, focusing on a profit-maximizing retailer under uncertain demand at the aggregate level, is significantly different from the three studies cited above in several aspects. First, a monopolistic market comprised of two segments (i.e., the offline and online segments), is incorporated in the modeling framework. Second, the quantity of a product to be allocated to each segment by the retailer is treated as a decision variable. Third, the demand in each segment is modeled as a continuous random variable. Fourth, the optimal allocation scheme is analytically determined under uncertain demand following a uniform probability distribution. Fifth, this study examines the impacts of changes in six key model parameters on retailer's profitability.

The three research questions that we attempt to address in this paper are specifically stated as follows: (i) What is the optimal scheme for a retailer to allocate K identical units of a product to the offline and online segments during a single selling season so that its expected total profit is maximized? (ii) What is the impact of changes in the unit salvage value of each segment on the retailer's profitability? (iii) What are the impacts of changes in the price sensitivities of each segment on retailer's profitability? We make the following basic assumptions while addressing these questions of strategic significance:

- (i) The demand of each of the offline and online segments is affected by the prices charged in both segments. (In general, this assumption reflects the reality of consumer behavior. For example, the higher price of a textbook in a brick-and-mortar bookstore may force a student to buy a less expensive copy from an online bookstore.)
- (ii) Consumers in each segment are well-informed of the prices in both segments, which are set at the beginning of the selling season.
- (iii) The demand of each segment is heterogeneous and follows a continuous probability distribution conditioned by the prices charged in both segments.

The rest of the paper is organized as follows. In the next section, we develop the retailer's profit functions and then formulate a mathematical programming model to find the optimal allocation scheme for the product. The third section presents six propositions that highlight the optimal allocation scheme and the impacts of changes in six key model parameters on the retailer's profitability. The fourth section reports the computational results of a simulation experiment conducted to numerically compare nine alternative allocation schemes and examine the impacts of the six parameters mentioned above. Finally, our study concludes in its fifth section with a summary of its contributions, limitations, and directions for future research.

MODEL DEVELOPMENT

Let us consider a monopolistic retailer that has K identical units of a product to be allocated to its brick-and-mortar and online stores during a single selling season. The offline and online market segments for the product are denoted as Segments 1 and 2, respectively. Several terms are defined below in formulating the retailer's profit functions:

y_i	the quantity of the product to be allocated to Segment i (a decision variable);
P_i	the price per unit of the product charged to Segment i ($P_i > 0$);
C_i	the cost per unit of the product ($C_i > 0$);
S_i	the salvage value per unit of the product disposed of after the selling season ($S_i > 0$);
d_i	the aggregate demand for consumers of Segment i ;
ω_i	the demand parameter of Segment i ($\omega_i > 0$);
$f(x \omega_i)$	the probability density function (p.d.f.) of d_i being a continuous random variable;
$E(d_i)$	the expected value of d_i ;
π_i	the retailer's profit yielded from Segment i ;
$E(\pi_i)$	the expected value of π_i ;
π	the retailer's total profit yielded from the entire two-segment market;
$E(\pi)$	the expected value of π .

The aggregate demand in a market segment is usually uncertain (e.g., Deng et al., 2008). Since consumers are assumed to be well-informed of the prices charged to Segments 1 and 2, they would take the prices into consideration while making their purchases. Hence, the demand of Segment i , d_i ($i = 1, 2$), is modeled as a random variable following a probability distribution conditioned by P_1 and P_2 . The p.d.f. of d_i takes the form of $f(x|\omega_i)$ and d_i is a continuous random variable.

For Segment i ($i = 1, 2$), if the demand (d_i) exceeds the quantity of the product allocated by the retailer (y_i), the profit (π_i) will equal the profit per unit multiplied by the number of units sold. On the other hand, if d_i is smaller than y_i , a portion of the allocated quantity, $y_i - d_i$, will be unsold and disposed of after the selling season at the unit salvage value, S_i . Hence, the retailer's profit yielded from segment i ($i = 1, 2$) is expressed as

$$\pi_i = \begin{cases} P_i d_i - C_i y_i + S_i (y_i - d_i), & d_i \leq y_i, \\ (P_i - C_i) y_i, & d_i > y_i. \end{cases} \quad (1)$$

As d_i is a continuous random variable, the expected profit yielded from Segment i is derived from (1) as follows:

$$E(\pi_i) = (P_i - S_i) \int_0^{y_i} x f_i(x|\omega_i) dx - (C_i - S_i) y_i \int_0^{y_i} f_i(x|\omega_i) dx + (P_i - C_i) y_i \int_{y_i}^{\infty} f_i(x|\omega_i) dx. \quad (2)$$

Following Azoury (1985), we model the aggregate demand d_i as a uniformly distributed random variable. The p.d.f. of d_i is assumed to take the following continuous form:

$$f(x|\omega_i) = \begin{cases} 1/\omega_i, & \text{if } 0 < x < \omega_i, \\ 0, & \text{if } x \geq \omega_i, \end{cases} \quad (3)$$

where $\omega_i > 0$. The expected demand of Segment i , based on (3), is given by

$$E(d_i) = \int_0^{\omega_i} \frac{x}{\omega_i} dx = \frac{\omega_i}{2}. \quad (4)$$

Expression (4) shows that the demand parameter, ω_i , equals twice the expected demand, $E(d_i)$, and thus serves as an indicator of the aggregate demand of Segment i .

Substituting (3) into (2) and carrying out the integrations yields

$$E(\pi_i) = (P_i - C_i)y_i - \frac{(P_i - S_i)}{2\omega_i}y_i^2, \quad \text{if } y_i < \omega_i; \quad (5)$$

$$E(\pi_i) = \frac{(P_i - S_i)}{2}\omega_i - (C_i - S_i)y_i, \quad \text{if } y_i \geq \omega_i. \quad (6)$$

The retailer's expected total profit from the entire two-segment market is expressed as

$$E(\pi) = E(\pi_1) + E(\pi_2). \quad (7)$$

Given K identical units of a product, which is to be allocated by the retailer during a single selling season for sale to the offline and online segments, we aim at finding the optimal quantity to be allocated to Segment i ($i = 1, 2$), y_i^* , to maximize the retailer's expected total profit. The problem may, therefore, be formulated as follows:

$$\begin{aligned} & \underset{y_1, y_2}{\text{Max}} [E(\pi_1) + E(\pi_2)] \\ \text{s.t.} \quad & y_1 + y_2 \leq K, \quad y_1, y_2 \geq 0. \end{aligned} \quad (8)$$

As it is assumed that the demand of each segment is dictated by both prices P_1 and P_2 , a linear demand model introduced in Huang et al. (2013) is employed in this study to model the expected demand of Segment i ($i = 1, 2$):

$$E(d_1) = \alpha_1 - \beta_1 P_1 + \beta_{12} P_2, \quad (9)$$

$$E(d_2) = \alpha_2 - \beta_2 P_2 + \beta_{21} P_1, \quad (10)$$

where, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_{12}, \beta_{21} > 0$.

For $i = 1, 2$, the constant α_i captures the part of the aggregate demand of Segment i that does not vary with prices; β_i measures the price sensitivity of the demand of Segment i to changes in the price charged in the same segment, P_i ; β_{12} measures the price sensitivity of the demand of Segment 1 to changes in the price charged in Segment 2, P_2 ; similarly, β_{21} measures the price sensitivity of the demand of Segment 2 to changes in P_1 .

OPTIMAL ALLOCATION OF PRODUCT QUANTITIES

Following the approach of Lee and Ng (2001), this study focuses on an optimal interior solution to model (8) such that $y_1^* + y_2^* < K$. In each of the following two cases, we first present a

solution to model (8) and then study the impacts of changes in the unit salvage value and price sensitivities of each segment on the retailer's optimal expected total profit.

Case 1: $y_i < \omega_i$ ($i = 1, 2$)

In this case, the quantity of the product allocated to Segment i is less than ω_i . Three propositions related to this case are introduced below, for which the proofs are found in the Appendix.

Proposition 1. Given $\sum_{i=1}^2 \frac{(P_i - C_i)}{(P_i - S_i)} \omega_i < K$, $P_i > C_i$ and $P_i > S_i$, the retailer's expected total profit

$E(\pi)$ reaches its maximal level $E^*(\pi) = \sum_{i=1}^2 E^*(\pi_i)$ at $y_i^* = \frac{(P_i - C_i)}{(P_i - S_i)} \omega_i$, where $E^*(\pi_i) = \frac{(P_i - C_i)^2}{2(P_i - S_i)} \omega_i$,

$i = 1, 2$.

According to Proposition 1, the retailer can maximize its expected total profit by allocating the quantity of the product to each segment at the optimal level. The optimal quantity allocated to Segment i , y_i^* , is determined by the price charged, the unit cost, the unit salvage value, and the expected demand of the segment (see expression (4)). The conditions of $P_i > C_i$ and $P_i > S_i$ indicate $y_i^* > 0$ ($i = 1, 2$).

Proposition 2. Given $P_i > C_i$ and $P_i > S_i$ ($i = 1, 2$) and all other things being equal, $E^*(\pi)$ is monotonically increasing in S_i .

Proposition 3. Given $P_i > C_i$ and $P_i > S_i$ ($i = 1, 2$) and all other things being equal:

- (i) $E^*(\pi)$ is monotonically decreasing in β_i ($i = 1, 2$).
- (ii) $E^*(\pi)$ is monotonically increasing in β_{ij} ($i, j = 1, 2; i \neq j$).

Proposition 2 shows, as expected, that profitability is enhanced as the salvage values become larger. Proposition 3 demonstrates that profitability is enhanced as the cross-price sensitivities among segments become larger and the segments' self-price sensitivities become smaller.

Case 2. $y_i \geq \omega_i$ ($i = 1, 2$)

In this case, the quantity of the product allocated to Segment i is more than or equal to ω_i . Three propositions are introduced for this case, for which the proofs are found in the Appendix.

Proposition 4. Given $(C_i - S_i) > 0$, $P_i + S_i - 2C_i > 0$, and $\sum_{i=1}^2 \omega_i \leq K$, the firm's expected total

profit $E(\pi)$ reaches its maximal level $E^*(\pi) = \sum_{i=1}^2 E^*(\pi_i)$ at $y_i^* = \omega_i$ for $i = 1, 2$, where

$$E^*(\pi_i) = \frac{(P_i + S_i - 2C_i)}{2} \omega_i.$$

Proposition 4 shows that for Case 2, the retailer should allocate the quantity of the product exactly equal to twice the expected demand of each segment to maximize its expected total profit (see expression (4)).

Proposition 5. *Given all other things being equal, $E^*(\pi)$ is monotonically increasing in S_i ($i = 1, 2$).*

Proposition 6. *Given all other things being equal:*

- (i) *For $P_i + S_i - 2C_i > 0$, $E^*(\pi)$ is monotonically decreasing in β_i ($i = 1, 2$).*
- (ii) *For $P_i + S_i - 2C_i > 0$, $E^*(\pi)$ is monotonically increasing in β_{ij} ($i, j = 1, 2; i \neq j$).*

Proposition 5 shows, as expected, that profitability is enhanced as the salvage values become larger. Proposition 6 demonstrates that profitability is enhanced as the cross-price sensitivities among segments become larger and the segments' self-price sensitivities become smaller.

NUMERICAL ILLUSTRATIONS

This section presents a simulation experiment to (i) compare the optimal allocation scheme with other eight alternative allocation schemes and (ii) explore the impacts of six model parameters on the retailer's expected total profit. For illustrative purposes, the experiment is conducted only for Case 1 discussed in the third section. Due to the empirical evidence that consumers are less price sensitive when they shop online (see Chu et al., 2008), we particularly choose the values of β_1 and β_2 such that $\beta_1 > \beta_2$. The base values of all the model parameters chosen for the experiment are given below:

$$\alpha_1 = 40000 \text{ units}, \alpha_2 = 30000 \text{ units}; \beta_1 = 45, \beta_2 = 35; \beta_{12} = 20, \beta_{21} = 15;$$

$$P_1 = \$550, P_2 = \$450; C_1 = \$350, C_2 = \$200; S_1 = \$150, S_2 = \$150.$$

Based on Proposition 1 developed for Case 1 (see the previous section), the optimal allocation scheme (y_1^*, y_2^*) and the corresponding expected total profit $E^*(\pi)$ are calculated and reported in Table 1. The expected demands of Segments 1 and 2 (determined by expressions (9) and (10), respectively) are also shown in the table. It is found that for the selected values of model parameters, the optimal allocation scheme ($y_1^* = 24250$ units, $y_2^* = 37500$ units) yields the highest expected total profit of \$7,112,500 in comparison to the other eight allocation schemes. The expected total profit for each of the other allocation schemes are computed using expression (5) noting from expression (4) that $\omega_i = 2 E(d_i); i = 1, 2$.

The impacts of changes in each of the six key parameters are investigated while all the other parameters are held constant at their aforementioned base values, and the computational results are reported in Table 2. As shown in the table, an increase in either of S_1 , S_2 , β_{12} , or β_{21} enhances retailer's profitability, whereas an increase in either β_1 or β_2 reduces the profitability. These findings are consistent with Propositions 2 and 3.

SUMMARY AND CONCLUSIONS

This paper tackles the problem of optimally allocating K identical units of a product by a monopolistic retailer to its brick-and-mortar and online stores under uncertain demand following a uniform probability distribution. A mathematical programming model is developed and then analytically solved to determine the optimal scheme of product allocation. In addition, the impacts of changes in the unit salvage value and the two types of price sensitivity for both offline and online segments are analytically investigated. A simulation experiment is conducted to

Table 1 Alternative schemes of quantity allocation and the resultant expected total profits

Prices (\$/unit)		Expected Demands (in units)		Allocation Schemes (in units)		Expected Total Profit
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1^* = 24250$	$y_2^* = 37500$	$E^*(\pi) = \$7,112,500$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 20000$	$y_2 = 37500$	$E(\pi) = \$7,038,016$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 30000$	$y_2 = 37500$	$E(\pi) = \$6,976,160$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 24250$	$y_2 = 30000$	$E(\pi) = \$6,925,000$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 24250$	$y_2 = 40000$	$E(\pi) = \$7,091,667$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 20000$	$y_2 = 30000$	$E(\pi) = \$6,850,515$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 30000$	$y_2 = 40000$	$E(\pi) = \$6,955,326$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 20000$	$y_2 = 40000$	$E(\pi) = \$7,017,182$
$P_1 = \$550$	$P_2 = \$450$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1 = 30000$	$y_2 = 30000$	$E(\pi) = \$6,788,660$

Table 2 The impacts of six key model parameters on the optimal expected total profit

Model Parameters	Expected Demands (in units)		Allocation Schemes (in units)		Expected Total Profit
$S_1 = \$130/\text{unit}$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1^* = 23095$	$y_2^* = 37500$	$E^*(\pi) = \$6,997,024$
$S_1 = \$170/\text{unit}$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1^* = 25526$	$y_2^* = 37500$	$E^*(\pi) = \$7,240,132$
$S_2 = \$130/\text{unit}$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1^* = 24250$	$y_2^* = 35156$	$E^*(\pi) = \$6,819,531$
$S_2 = \$170/\text{unit}$	$E(d_1) = 24250$	$E(d_2) = 22500$	$y_1^* = 24250$	$y_2^* = 40179$	$E^*(\pi) = \$7,447,321$
$\beta_1 = 35$	$E(d_1) = 29750$	$E(d_2) = 22500$	$y_1^* = 29750$	$y_2^* = 37500$	$E^*(\pi) = \$7,662,500$
$\beta_1 = 55$	$E(d_1) = 18750$	$E(d_2) = 22500$	$y_1^* = 18750$	$y_2^* = 37500$	$E^*(\pi) = \$6,562,500$
$\beta_2 = 25$	$E(d_1) = 24250$	$E(d_2) = 27000$	$y_1^* = 24250$	$y_2^* = 45000$	$E^*(\pi) = \$8,050,000$
$\beta_2 = 45$	$E(d_1) = 24250$	$E(d_2) = 18000$	$y_1^* = 24250$	$y_2^* = 30000$	$E^*(\pi) = \$6,175,000$
$\beta_{12} = 15$	$E(d_1) = 22000$	$E(d_2) = 22500$	$y_1^* = 22000$	$y_2^* = 37500$	$E^*(\pi) = \$6,887,500$
$\beta_{12} = 25$	$E(d_1) = 26500$	$E(d_2) = 22500$	$y_1^* = 26500$	$y_2^* = 37500$	$E^*(\pi) = \$7,337,500$
$\beta_{21} = 10$	$E(d_1) = 24250$	$E(d_2) = 19750$	$y_1^* = 24250$	$y_2^* = 32917$	$E^*(\pi) = \$6,539,583$
$\beta_{21} = 20$	$E(d_1) = 24250$	$E(d_2) = 25250$	$y_1^* = 24250$	$y_2^* = 42083$	$E^*(\pi) = \$7,685,417$

numerically demonstrate the superiority of the optimal allocation scheme and the impacts of six model parameters on retailer's profitability.

Our exploratory study reveals several directions for future research. First, this study provides the analytical solutions to the problem of optimizing product allocation under uncertain demand following a uniform probability distribution. The demand following other probability distributions could be incorporated in the modeling framework. Second, in this study we focus on a retailer in a monopolistic market. Developing a model for a competitive environment would be a plausible extension. Third, while the optimal allocation scheme is determined in this study for a single product, a multiple-product scenario would be explored in the future. In addition, prices were assumed to have been determined exogenously. Treating prices as decision variables would offer an interesting direction for future research.

APPENDIX

Derivation of expression (5)

Given $y_i < \omega_i$, substituting (3) into (2) and carrying out the three integrations yield:

$$\begin{aligned}
 E(\pi_i) &= (P_i - S_i) \int_0^{y_i} \frac{x}{\omega_i} dx - (C_i - S_i) y_i \int_0^{y_i} \frac{1}{\omega_i} dx + (P_i - C_i) y_i \int_{y_i}^{\omega_i} \frac{1}{\omega_i} dx \\
 &= \frac{(P_i - S_i)}{2\omega_i} y_i^2 - \frac{(C_i - S_i)}{\omega_i} y_i^2 + (P_i - C_i) y_i - \frac{(P_i - C_i)}{\omega_i} y_i^2 \\
 &= (P_i - C_i) y_i - \frac{(P_i - S_i)}{2\omega_i} y_i^2.
 \end{aligned} \tag{A.1}$$

Derivation of expression (6)

Given $y_i \geq \omega_i$, substituting (3) into (2) and carrying out the three integrations yield:

$$E(\pi_i) = (P_i - S_i) \int_0^{\omega_i} \frac{x}{\omega_i} dx - (C_i - S_i) y_i \int_0^{\omega_i} \frac{1}{\omega_i} dx = \frac{(P_i - S_i)}{2} \omega_i - (C_i - S_i) y_i. \tag{A.2}$$

Proof of Proposition 1

Given the objective function of model (8) where $E(\pi_i)$ takes expression (5), the first-order necessary condition for optimality implies that

$$\frac{\partial E(\pi)}{\partial y_i} = (P_i - C_i) - \frac{(P_i - S_i)}{\omega_i} y_i = 0, \quad i = 1, 2. \tag{A.3}$$

Solving the pair of equations (A.3) simultaneously, we obtain

$$y_i^* = \frac{(P_i - C_i)}{(P_i - S_i)} \omega_i, \quad i = 1, 2. \quad (\text{A.4})$$

The total optimal quantity of the product allocated to Segments 1 and 2 is given by

$$\sum_{i=1}^2 y_i^* = \sum_{i=1}^2 \frac{(P_i - C_i)}{(P_i - S_i)} \omega_i. \quad (\text{A.5})$$

A feasible interior solution to model (8) requires $\sum_{i=1}^2 \frac{(P_i - C_i)}{(P_i - S_i)} \omega_i < K$. Substituting y_i^* ($i = 1, 2$) into expressions (5) and (7) yields

$$E^*(\pi) = \sum_{i=1}^2 \frac{(P_i - C)^2}{2(P_i - S_i)} \omega_i. \quad (\text{A.6})$$

We now prove that the retailer's expected total profit $E(\pi)$ reaches its maximal level, $E^*(\pi)$, at y_i^* ($i = 1, 2$).

$$\text{In our case, } \frac{\partial E^2(\pi)}{\partial y_i^2} = (-1) \frac{(P_i - S_i)}{\omega_i} < 0, \quad \frac{\partial E^2(\pi)}{\partial y_1 \partial y_2} = 0 \quad \text{and} \quad \frac{\partial E^2(\pi)}{\partial y_2 \partial y_1} = 0.$$

$$\text{Hence, } \left(\frac{\partial E^2(\pi)}{\partial y_1 \partial y_2} \right) \left(\frac{\partial E^2(\pi)}{\partial y_2 \partial y_1} \right) - \left(\frac{\partial E^2(\pi)}{\partial y_1^2} \right) \left(\frac{\partial E^2(\pi)}{\partial y_2^2} \right) < 0.$$

The sufficient condition for $E(\pi)$ to reach its maximal level $E^*(\pi)$ at y_i^* ($i = 1, 2$) is satisfied. ■

Proof of Proposition 2

Consider $E^*(\pi) = \sum_{i=1}^2 E^*(\pi_i)$, where $E^*(\pi_i) = \frac{(P_i - C_i)^2}{2(P_i - S_i)} \omega_i$ (see Proposition 1). We obtain the first partial derivative with respect to S_i :

$$\frac{\partial E^*(\pi)}{\partial S_i} = \frac{(P_i - C_i)^2}{2(P_i - S_i)^2} \omega_i. \quad (\text{A.7})$$

Given $P_i > C_i$ and $P_i > S_i$, $\frac{\partial E^*(\pi)}{\partial S_i} > 0$ and hence, $E^*(\pi)$ is monotonically increasing in S_i . ■

Proof of Proposition 3

Consider $E^*(\pi) = \sum_{i=1}^2 E^*(\pi_i)$, where $E^*(\pi_i) = \frac{(P_i - C_i)^2}{2(P_i - S_i)} \omega_i$ and $\omega_i = 2(\alpha_i - \beta_i P_i + \beta_{ij} P_j)$ for $i, j = 1, 2$; $i \neq j$. We first obtain the first partial derivative with respect to β_i :

$$\frac{\partial E^*(\pi)}{\partial \beta_i} = \frac{(P_i - C_i)^2}{(P_i - S_i)} (-P_i). \quad (\text{A.8})$$

Given $P_i > C_i$ and $P_i > S_i$, $\frac{\partial E^*(\pi)}{\partial \beta_i} < 0$ and hence, $E^*(\pi)$ is monotonically decreasing in β_i .

We also obtain the first partial derivative with respect to β_{ij} :

$$\frac{\partial E^*(\pi)}{\partial \beta_{ij}} = \frac{(P_i - C)^2}{(P_i - S_i)} P_j.$$

Given $P_i > C_i$ and $P_i > S_i$, $\frac{\partial E^*(\pi)}{\partial \beta_{ij}} > 0$ and hence, $E^*(\pi)$ is monotonically increasing in β_{ij} . ■

Proof of Proposition 4

In Case 2, the focal firm's expected profit from segment i , $E(\pi_i)$, takes expression (6), which indicates that given $(C_i - S_i) > 0$,

$$\frac{(P_i - S_i)}{2} \omega_i - (C_i - S_i) y_i \leq \frac{(P_i - S_i)}{2} \omega_i - (C_i - S_i) \omega_i \text{ holds for any } y_i \geq \omega_i \ (i = 1, 2).$$

Hence,

$$\sum_{i=1}^2 \frac{(P_i - S_i)}{2} \omega_i - (C_i - S_i) y_i \leq \sum_{i=1}^2 \frac{(P_i - S_i)}{2} \omega_i - (C_i - S_i) \omega_i = \sum_{i=1}^2 \frac{(P_i + S_i - 2C_i)}{2} \omega_i. \quad \blacksquare$$

Proof of Proposition 5

Consider $E^*(\pi) = \sum_{i=1}^2 E^*(\pi_i)$, where $E^*(\pi_i) = \frac{(P_i + S_i - 2C_i)}{2} \omega_i$ (see Proposition 4). We obtain the first partial derivative with respect to S_i :

$$\frac{\partial E^*(\pi)}{\partial S_i} = \frac{1}{2} \omega_i > 0. \quad (\text{A.9})$$

Hence, $E^*(\pi)$ is monotonically increasing in S_i . ■

Proof of Proposition 6

Consider $E^*(\pi) = \sum_{i=1}^2 E^*(\pi_i)$, where $E^*(\pi_i) = \frac{(P_i + S_i - 2C_i)}{2} \omega_i$ and $\omega_i = 2(\alpha_i - \beta_i P_i + \beta_{ij} P_j)$ for $i, j = 1, 2; i \neq j$. We first obtain the first partial derivative with respect to β_i :

$$\frac{\partial E^*(\pi)}{\partial \beta_i} = (P_i + S_i - 2C_i)(-P_i). \quad (\text{A.10})$$

- (i) If $P_i + S_i - 2C_i > 0$, then $\frac{\partial E^*(\pi)}{\partial \beta_i} < 0$ and hence, $E^*(\pi)$ is monotonically decreasing in β_i .

We also obtain the first partial derivative with respect to β_{ij} :

$$\frac{\partial E^*(\pi)}{\partial \beta_{ij}} = (P_i + S_i - 2C_i)P_j. \quad (\text{A.11})$$

- (ii) If $P_i + S_i - 2C_i > 0$, then $\frac{\partial E^*(\pi)}{\partial \beta_{ij}} > 0$ and hence, $E^*(\pi)$ is monotonically increasing in β_{ij} . ■

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