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An economic production lot size model in an imperfect production system with a Markov process and shortage backordering

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ABSTRACT

This paper studies optimal lot size for an imperfect production system in which the shortages are allowed and the deterioration process is characterized by a two-state discrete-time Markov chain. We show that there exists a unique optimal lot size to minimize the expected total relevant cost. In addition, bounds for the optimal lot size are provided to develop the solution procedure. Finally, a numerical example is given to illustrate the results. Sensitivity analysis and managerial insights are also provided.

KEYWORDS: Lot sizing, Imperfect production system, Markov chain; Shortages

INTRODUCTION

Considerable attention has also been paid to EPQ model in the presence of deteriorating production process. The effect of process deterioration on the lot size was initially studied by Rosenblatt & Lee (1986) and Porteus (1986). Specifically, Porteus (1986) considered an EOQ model with deteriorating process and assume that all items produced are defective when the system is out-of-control. However, Rosenblatt & Lee (1986) assumed that a proportion of the items produced are defective once the production process is out-of-control. In addition, Porteus (1986) showed that the optimal lot size is smaller than that of the traditional EOQ model. However, as we will see later on, this result may not be true when the maintenance cost is high as compared to the rework cost incurred by defective items. More recently, Hou et al. (2015) extended Porteus (1986) to present the optimal lot size model for defective items with a constant probability when the system is out-of control and taking maintenance cost into account. Maddah et al. (2010) considered an inventory system within the economic order quantity model framework under random supply where imperfect items are removed from inventory. Sana (2010) considered an economic production lot size model in an imperfect production system and assumed that a certain percent of total product is defective in the out-of-control state. Yoo et al. (2012) examined an imperfect production and inspection system with customer return and defective disposal. Other related studies considering imperfect quality production process include Chen & Kang (2010), Goyal & Cárdenas-Barrón (2002), Khan et al. (2011), Lee and Kim (2014), Maddah & Jaber (2008), Pal et al. (2012-2013), Papachristors & Konstantaras (2006), Salameh & Jaber (2000), and Sarkar (2012).

All of the above-mentioned models assume that shortages are not permitted to occur. However, in many practical situations, stockout is unavoidable due to various uncertainties. Therefore, the main purpose of this paper is to generalize the traditional EPQ model to assume that shortages are allowed in an imperfect production system with a Markov process. In addition, we focus on investigating the effects of system deterioration on the optimal lot size with maintenance actions when the defective items can be reworked and shortages are completely backlogged. We show that there exists a unique optimal lot size such that the expected total relevant cost is minimized. Bounds for the optimal lot size are also provided to develop the solution procedure. The analyses show that our model is a generalization of the models in current literatures. Specifically, the numerical results in this paper show that our model always results in better performance.

NOTATION AND ASSUMPTIONS

To develop an EPQ model for the deteriorating production system, the following notation and assumptions are adopted:

Notation

d demand rate,
p production rate, $p > m$,
K setup cost for each production run,
h holding cost per unit per unit time,
c_r rework cost for a defective item,
R maintenance cost for restoring the system from out-of-control back to in-control,
π the backordering cost per unit
q the probability that the system from in-control state shifts to out-of-control state,
$\overline{q}$ the probability that the system stays in-control state during the production of an item and $\overline{q}=1\text{-}q$,
θ the percentage of defective items when the process is in the out-of-control state,
$I_{\max}$ the maximum on-hand inventory level,
b the maximum backordering quantity in units,
N number of defective items produced in a production run,
T cycle time of a production run,
Q lot size for each production run,
$TC(Q,b)$ the expected total relevant cost per unit time for lot size Q and maximum backordering quantity b,

Assumptions

We develop the model with the following assumptions:

1. We suppose that the operating condition of the production system at any time can be classified into one of the two states, i.e., in-control and out-of-control.
2. When the system is in the in-control state, it may either shift to the out-of-control state with probability q or stay in the in-control state with probability $1\text{-}q$ during the production of an item. Once the system shifts to the out-of-control state, it remains in this state until the end of a production run. That is, the deteriorating process can be modeled by a two-state discrete-time Markov chain.
3. After the production of a lot of size Q, the production process is inspected to reveal the state of the process. If the process is out-of-control then it is restored to the in-control state with restoration cost R for the next production run.

4. Due to manufacturing variability, if the production process is out-of-control, then it will lead to an item produced is defective with a fixed proportion θ not rather than all items produced are defective as in Porteus (1986).
5. The defective items will eventually be reworked with cost c_r such that the capacity of the production system is not affected.
6. Shortages are allowed and are completely backlogged.

THEORETICAL MODEL

The expected total relevant cost per unit time composed of setup costs, holding costs, backorder costs, restoration cost, and rework costs. These costs are evaluated as follows.

(1) Setup cost: the setup cost per unit time is given by

$$\frac{K}{T}=\frac{Kd}{Q} \tag{1}$$

(2) Holding cost: During each cycle, holding cost will occur during T_2 and T_3. Note that $T_2 = I_{max}/(p\text{-}d)$ and $T_3 = I_{max}/d$. So the average inventory over the cycle is the area under the inventory triangle, divided by T. Denoting this by $\bar{I}$, we have

$$\bar{I}=\frac{1}{T}\left\{\frac{[Q(1-d/p)-b]^2}{2d(1-d/p)}\right\}$$

$$=\frac{[Q(1-d/p)-b]^2}{2Q(1-d/p)} \tag{2}$$

Thus, according to Eq. (2), the holding cost per unit time is given by

$$h\bar{I}=\frac{h[Q(1-d/p)-b]^2}{2Q(1-d/p)} \tag{3}$$

(3) Backordering cost: The backordering cost will occur during T_1 and T_4. Note that $T_1=b/(p\text{-}d)$ and $T_4=b/d$. So that the average backorder level is given by

$$\bar{B}=\frac{1}{T}\left[\frac{b^2}{2d(1-d/p)}\right]$$

$$=\frac{b^2}{2Q(1-d/p)} \tag{4}$$

Thus, according to Eq. (4), the backordering cost per unit time is given by

$$\pi\bar{B}=\frac{\pi b^2}{2Q(1-d/p)} \tag{5}$$

(4) Restoration cost: The restoration cost occurs only when the production process is out-of-control at the end of a production run for lot size Q. Thus, the expected restoration cost becomes $R(1-\bar{q}^Q)$. Hence, the restoration cost per unit time is given by

$$\frac{R(1-\bar{q}^Q)}{T}=\frac{dR(1-\bar{q}^Q)}{Q} \tag{6}$$

(5) Rework cost: To obtain the expected rework cost, we need to compute the expected number of defective items in a lot. Let $\bar{q}=1-q$ and in a lot of size Q, the probability distribution of number of items produced in the in-control state, X, is

$$\Pr\{X=j\}=\begin{cases}\bar{q}^j q, & 0\le j<Q\\ \bar{q}^Q, & j=Q\end{cases} \tag{7}$$

Then, the expected value of X is given by

$$E(X)=q\sum_{j=1}^{Q-1} j\bar{q}^j+Q\bar{q}^Q$$

$$=\sum_{j=1}^{Q}\bar{q}^j \tag{8}$$

So that the number of defective items in a lot of size Q becomes $N=\theta(Q-X)$. Hence, the expected value of N is

$$E(N)=\theta(Q-\sum_{j=1}^{Q}\bar{q}^j) \tag{9}$$

Thus, according to Eq. (9), the expected rework cost per unit time is given by

$$\frac{c_r E(N)}{T} = c_r d\theta - \frac{d}{Q}\left(c_r \theta \sum_{j=1}^{Q} \bar{q}^{\,j}\right) \tag{10}$$

Let *TC*(*Q*,*b*) be the expected total relevant cost per unit time composed of setup cost, inventory holding cost, backordering cost, restoration cost, and rework cost. Then *TC*(*Q*,*b*) is given as follows.

$$TC(Q,b) = \frac{dK}{Q} + \frac{h[Q(1-d/p)-b]^2}{2Q(1-d/p)} + \frac{\pi b^2}{2Q(1-d/p)} + c_r d\theta + \frac{d}{Q}\left[R(1-\bar{q}^{\,Q}) - c_r\theta \sum_{j=1}^{Q} \bar{q}^{\,j}\right] \tag{11}$$

Notice that the *TC*(*Q*,*b*) is convex with respect to *b*. Thus, taking the first derivate of the *TC*(*Q*,*b*) with respect to *b* and equating it to zero will yield the optimal backordering quantity as follows.

$$b^* = \frac{hQ(1-d/p)}{h+\pi} \tag{12}$$

Substituting Eq. (12) unto Eq. (11) yields the following expression of *C*(*Q*).

$$C(Q) = \frac{dK}{Q} + \frac{hQ(1-d/p)}{2}\left(\frac{\pi}{h+\pi}\right) + c_r d\theta + \frac{d}{Q}\left[R(1-\bar{q}^{\,Q}) - c_r\theta \sum_{j=1}^{Q} \bar{q}^{\,j})\right] \tag{13}$$

So far, we have appropriately established mathematical model to investigate the lot size for a deteriorating production system. The problem is then to find the optimal lot size Q^* that leads to the minimum of $C(Q)$. In addition, based on Eq. (13), following observations should be mentioned:

(1) When q = 0, Eq. (13) reduces to the case that the production system does not deteriorate and is always in the in-control state and items produced are perfect quality.

(2) If π approaches to infinity, Eq. (13) becomes the expected total relevant cost of no-shortage-allowed case. Thus, Eq. (13) will reduce to Eq. (8) in Hou et al. (2015). Therefore, the model in this paper is an extension of Hou et al. (2015).

(3) Combining the arguments of (1) and (2), Eq. (13) will reduce to the traditional EPQ model. In other words, the traditional EPQ model is a special case of the model proposed in this paper when the system deterioration, defective items, and shortages are all not considered.

(4) Note that as π approaches to infinity, it is equivalent to instantaneous production of the entire lot, the Eq. (13) reduces to the EOQ model with an imperfect production process.

(5) When R = 0, Eq. (13) reduces to the case that the restoration cost is not considered.

(6) When θ=1, Eq. (13) reduces to the case that all items produced are defective as the production process is in an out-of-control state.

Combining the arguments of (2) and (4)–(6), Eq. (13) will reduce to the model in Porteus (1986). Therefore, the proposed model in this paper is also an extension of Porteus (1986).

NUMERICAL RESULTS

In order to illustrate the performance of our proposal model, let us consider an imperfect production system with the following data: p = 1500 units/year, d = 1000 units/year, K = \$600/cycle, h = \$8/unit/year, π = \$10/unit/year, c_r = \$5/unit, R = \$200/cycle, θ = 0.75, and q = 0.1. Using the above parameters, it can be verified that β is positive (β=166.25). Applying the algorithm described above, we find that optimal lot size Q^* = 1017.07, b^* =150.677, and expected total relevant cost per unit time $TC(Q^*,b^*)$ = \$5256.775. In addition, we show that optimal lot size Q* is larger than the traditional EPQ model (that is, $Q^* > Q_{tr}$ = 670.82) for the case of $\beta > 0$. It is unlike the results in previous works, e,g, see Maddah et al. (2010), Porteus (1986), and Rosenblatt & Lee (1986). The reason is that they only derive an approximately optimal lot size and do not take the restoration cost into account properly. However, the optimal lot size Q* may be larger than the traditional EPQ when the restoration cost is considered and relatively high as compared to the rework cost incurred by defective items.

CONCLUSIONS

In this paper, we propose a generalized production lot size model with system deterioration, imperfect quality and shortage backordering. The proposed model can be considered as a good extension of the previous literatures. We show that there exists a unique optimal lot size such that the expected total relevant cost is minimized no matter whether q is small or not. In addition, the bounds of optimal lot size are provided so that the bisection method can be used to locate optimal lot size easily. The analyses show that our model is a generalization of the models in current literatures. The numerical results in this paper show that our model always results in better performance. Specifically, significant cost savings can be achieved by considering shortages backordered in our proposed model.

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