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Bermudan put option pricing under risk aversion in bilateral gamma models

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ABSTRACT

We consider a Bermudan put option holder seeking to maximize her discounted expected payoff subject to a maximum allowable payoff variance. Under a bilateral gamma asset price model, we show that the option's price is the optimal value for a linear maximization problem for which we establish a lower bound.

KEYWORDS: Bermudan put options, exercise policy, option pricing, payoff variance, bilateral gamma processes

1 INTRODUCTION

Research in American or Bermudan financial options has as its principal goal, to determine the exercise strategy (policy) which maximizes that sort of option's expected discounted payoff (under an explicit model for how the price of a primary asset evolves). Usually, the optimal exercise strategy is such that one should exercise the option as soon as the asset's price crosses a threshold to enter a special subset of the state space called the *optimal exercise region*. The value function of the optimal exercise policy-in a frictionless market without the possibility of arbitrage-corresponds to that option's price. In other words, by evaluating an option's optimal exercise policy, we effectively price the option.

The *payoff uncertainty* associated with an exercise policy is measured as the variance of the payoff yielded by that policy, and is a reflection of the risk associated with the exercise policy. By focusing only on how to maximize an option's expected payoff, state-of-the-art option pricing models implicitly assume that that option holders do not care about payoff uncertainty. That is, option holders are typically considered to be risk-neutral. For a sample of recent articles on risk neutral pricing procedures for a variety of financial options, we point the reader to Hurd and Zhou (2010), Bjerksund and Stensland (2014), Mariani et al. (2015), and Leccadito et al. (2016). However, in practice, option holders do not unconditionally seek to maximize expected payoff, but also pay attention to the uncertainty surrounding the expected payoff linked to any exercise policy under consideration. Put simply, option holders have finite payoff risk tolerance. In this paper, we attempt to price a Bermudan put option without the idealistic assumption that option holders have no sensitivity to payoff risk. A Bermudan put option offers the holder the right (without the obligation) to sell an asset for a deterministic strike price at any one of equally-spaced preselected dates within a certain time horizon. Hence, our problem setting is such that the option holder seeks a payoff-maximizing exercise policy for her option, but is only willing to search among the class of policies whose members have a payoff variance that does not exceed a pre-specified value of her choice. We call this pre-specified variance upper bound the option holder's *payoff uncertainty budget*.

The rest of the paper is organized as follows. Section 2 lays the foundation of our asset price model principally by spelling out important features of the probability distribution which characterizes our asset's random price fluctuations. In Section, 3 we solve the pricing problem via linear programming, and approximate a lower bound. Section 4 shows the results of a numerical study on some problems and Section 5 concludes the paper and suggests avenues for further exploration.

2 ANALYTICAL MODEL

2.1 Asset price models

As a basis for modeling randomly-evolving returns of asset prices, the normal distribution is widespread. Indeed, academic research on financial options took off with the analysis of European options by Black and Scholes (1973) who used geometric Brownian motion (GBM) as a model for their underlying asset. In the GBM model, the natural logarithm of an asset's infinitesimal random price increments follows a normal distribution. However, the light-tailed structure of the normal distribution usually makes GBM a poor modeling distribution for empirical financial data. To this effect, researchers have advanced new processes based on alternative distributions. See for instance discrete mixtures of normal distributions (Kon, 1984), the variance gamma distribution (Madan and Seneta, 1990), the truncated Lévy flight (Mantegna and Stanley, 1994), and the CGMY distribution (Carr et al., 2002).

More recently, Küchler and Tappe (2008) have proposed and empirically studied the bilateral gamma (BG) distribution (i.e., difference of two identical gamma distributions) as the basis for asset price fluctuations. Based on this price evolution model, they proposed a method to price the European call option. To justify their entire framework, they advanced the following three arguments for the attractiveness of the BG distribution. First, the BG distribution can be tuned by up to four parameters thereby making it highly amenable to empirical data fitting. Second, path trajectories from processes based on the BG distribution possess a higher level of realism because they have an infinite number of jumps on any time interval. Finally, the BG distribution has analytically simple Lévy characteristics. For these reasons, we also adopt the BG distribution as the modeling distribution for our asset price movements.

2.2 Moment-generating function of the sum of increments of bilateral gamma processes

Suppose we have a Poisson counting process where given some constant $\lambda > 0$, the value $\lambda \Delta t$ designates the number of new events that we can expect to observe in $\Delta t > 0$ time units. The probability distribution of the time that will elapse until we observe the next $\alpha > 0$ Poisson events is described by the gamma distribution $\Gamma(\alpha;\lambda\,\Delta t)$ with shape parameter α and rate parameter $\lambda\,\Delta t$. Define U^+ to be a $\Gamma(\alpha^+;\lambda^+)$ distribution, and U^- to be a $\Gamma(\alpha^-,\lambda^-)$ distribution where $\alpha^+ > 0$, $\alpha^- > 0$, $\lambda^+ > 0$, $\lambda^- > 0$. The random variable $X = U^+ - U^-$ follows a BG distribution described by $\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-) \equiv \Gamma(\alpha^+, \lambda^+) * \Gamma(\alpha^-, -\lambda^-)$. To model the randomness of the logarithm of stock prices, Küchler and Tappe (2008) introduced a continuous-time process $X_t\colon t \geq 0$ whose infinitesimal increments are drawn from identical BG distributions. They called this process the bilateral gamma process, and proved that it an infinitely divisible Lévy process. That is, if $X_t$is a continuous-time BG process, for $t \in [0,T]$, and $0 \leq \tau_1 < t_1 < \tau_2 < t_2 \leq T$, then $X_{t_1} - X_{\tau_1}$ and $X_{t_2} - X_{\tau_2}$are independent BG random variables where $X_{t_1} - X_{\tau_1} \sim \Gamma\big(\alpha^+(t_1 - \tau_1), \lambda^+; \alpha^-(t_1 - \tau_1), \lambda^-\big)$ and $X_{t_2} - X_{\tau_2} \sim \Gamma\big(\alpha^+(t_2 - \tau_2), \lambda^+; \alpha^-(t_2 - \tau_2), \lambda^-\big)$. In other words, non-overlapping increments of BG processes are independent BG random variables.

Now, let $N \in \aleph^+$ such that $n \in \{0, \dots, N\}$, and suppose we also have a continuous time horizon $[0,T]$, that is split into N equal periods with demarcation dates labeled $n = 0, n = \Delta t, n = 2\Delta t, \dots, n = N\Delta t = T$. Due to the Lévy property, we obtain$\sum_{i=1}^{n} X_{i\Delta t} - X_{(i-1)\Delta t} = X_{n\Delta t} = X_{n\Delta t} - X_0 = n\Delta t X \sim \Gamma\,(\alpha^+ n\Delta t, \lambda^+; \alpha^- n\Delta t, \lambda^-)$ for $n \in \{1, \dots, N\}$, and $X_0 \equiv 0$. The parameters $\lambda^+$and λ^- can be interpreted as Δt intensities of two independent Poisson processes that collectively determine the structure of the density function of $X_{n\Delta t}$. By scaling λ^+, λ^- such that $\Delta t = 1$, we notice that the moment-generating function (MGF) of $X\Delta t$ is $\mathbf{E}\big[e^{\theta X \Delta t}\big] = \left(\frac{\lambda^+}{\lambda^+ - \theta}\right)^{\alpha^+ \Delta t} \left(\frac{\lambda^-}{\lambda^- + \theta}\right)^{\alpha^- \Delta t} = \left(\frac{\lambda^+}{\lambda^+ - \theta}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + \theta}\right)^{\alpha^-}$ for $\theta \in \Re$ and$\lambda^+ > \theta$. Therefore, we obtain

$$\mathbf{E}\big[e^{\theta \sum_{i=1}^{n} X_i - X_{i-1}}\big] = \mathbf{E}\big[e^{\theta X_n}\big] = \mathbf{E}\big[e^{\theta n X}\big] = \left(\left(\frac{\lambda^+}{\lambda^+ - \theta}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + \theta}\right)^{\alpha^-}\right)^{n} \tag{1}$$

as the MGF of $\sum_{i=1}^{n} X_i - X_{i-1} = X_n$. Equation (1) is directly relevant to the option pricing problem especially for the case where $\theta = 1$.

2.3 Problem setup

Consider a non-dividend-paying asset whose price is modeled as a real-valued *exponential bilateral gamma process* $Y_t = y_0 e^{X_t}$ on a stochastic basis $(\Omega, F, \{F\}_{t \geq 0}, \mathbf{P})$ satisfying the usual conditions where $X_t{:}\, t \in [0,T]$ is a bilateral gamma process with initial value $X_0 = x_0 = 0$, and $y_0 > 0$ is the (deterministic) initial value of Y_t. To keep our model tractable, we impose the restriction $\alpha^+ = \alpha^- = 1$. The asset's price $Y_t{:}\, t \in [0,T]$ is observable in real time but the holder owns a (Bermudan) put option which can only be exercised at equally-spaced dates $n \in \{0, \dots, N\}$.

Date N is the option's maturity (expiration) date, and Y_t is totally adapted to F_n such that at any exercisable date $n \in \{0, \dots, N\}$, the holder has an observed sequence of price realizations represented by $y_1, \dots, y_n$ and another sequence of unobserved random prices $Y_{n+1}, \dots, Y_N$. An exercise policy π is a control rule encoded as a function of the form $a_n^{\pi}{:}\, \Re \to \{0,1\}$ for $n \in \{0, \dots, N\}$ such that

$$a_n^{\pi}(y_n) = \begin{cases} 1, & \mathit{if\ based\ on\ the\ asset\ price\ } y_n \mathit{\ at\ date\ } n, \mathit{one\ should\ immediately\ exercise} \\ 0, & \mathit{if\ based\ on\ the\ asset\ price\ } y_n \mathit{\ at\ date\ } n, \mathit{one\ should\ not\ exercise} \end{cases}$$

The holder makes the decision to exercise the option based only on asset price information y_n that is available up to and including date n. Hence, the stochastic evolution of Y_t means that to each arbitrary exercise policy π is associated a discrete random variable $\tilde{n}$ distributed on the support $\{0, \ldots, N\}$ such that $\Pr(\tilde{n} = n)$ represents the probability of exercising the option exactly at date $n \in \{0, \ldots, N\}$. If the option is exercised at date n, then $a_n^{\pi}(y_n) = 1$ and the holder receives an immediate payoff exercise value of

$$g(n) \equiv (k(n) - y_n)^+ \equiv max\{k(n) - y_n, 0\} \text{ for } n \in \{0, \ldots, N\} \tag{2}$$

where $k(n) \equiv k_0(1 - e^{wn}) + mn \geq 0$ is the strike price at date $n \in \{0, \ldots, N\}$ and $k_0 > 0, m \geq 0, w \leq \frac{1}{N} \ln\left(1 + \frac{mN}{k_0}\right)$ are constants. The benefit of designing $k(n)$ in the above manner is that we can change its structure by appropriately selecting m and w. For instance $k(n)$ is; (i) k_0 for $n \geq 1$ if $m = 0, w = -\infty$, (ii) nondecreasing if $m \geq 0, w \leq 0$, (iii) concave parabolic if $m > 0, 0 < w \leq \frac{1}{N} \ln\left(1 + \frac{mN}{k_0}\right)$, and (iv) zero if $m = 0, 0 \leq w \leq \frac{1}{N} \ln\left(1 + \frac{mN}{k_0}\right)$.

At the terminal date $n = N$, the option expires (matures) and there is no reason to postpone exercising. Hence, the exercise decision takes place before, or at the terminal date i.e., $\sum_{i=1}^{N} a_i^{\pi}(y_i) = 1$. Define $\mathbf{E}[G^{\pi}(y_0)]$ to be the discounted expected payoff that the holder will receive by pursuing some arbitrary exercise policy π inducing a *random* exercise date $\tilde{n} \in \{0, \ldots, N\}$ given an initial stock price of y_0. The holder's problem is to select π such that she maximizes $\mathbf{E}[G^{\pi}(y_0)]$ without letting the variance of $G^{\pi}(y_0)$ (which we denote by $\mathbf{V}[G^{\pi}(y_0)]$) exceed a pre-specified payoff uncertainty budget $\sigma_{max}^2 > 0$. Of course, the market should be efficient enough such that arbitrage opportunities are non-existent otherwise the holder has a structural advantage over the other market participants. To summarize, the option holder is looking for an exercise policy π^* to solve

$$G^{\pi^*}(y_0) \equiv max_{\mathbf{V}[G^{\pi}(y_0)] \leq \sigma_{max}^2} \mathbf{E}[G^{\pi}(y_0)] \tag{3}$$

where $G^{\pi^*}(y_0)$ is the option's price from the standpoint of date $n = 0$. In the next section, we determine the optimal exercise policy for Problem (3) (as well as the option's price) via the solution to a linear optimization problem.

3 DETERMINING THE OPTIMAL EXERCISE POLICY AND THE OPTION'S PRICE

We begin with the following restriction so as to ensure an arbitrage-free market.

Proposition 1. *To guarantee the condition of zero market arbitrage (i.e., to ensure that Y_t is a local $\mathbf{P}$-martingale), it is necessary that for $n \in \{0, \ldots, N\}$*

$$\mathbf{E}[e^{X_n}] = \left(\left(\frac{\lambda^+}{\lambda^+ - 1}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + 1}\right)^{\alpha^-}\right)^n = 1 \tag{4}$$

where $\lambda^+ > 1$.

Proof. See Appendix A.

Given a discount factor $0 < \beta \leq 1$, the value

$$\mathbf{E}[u_n] \equiv \beta^n \mathbf{E}[(k(n) - Y_n)^+] = \begin{cases} \beta^n (k_0(1-e^{wn}) + mn - y_0 \mathbf{E}[e^{X_n}]), & if\ k_0(1-e^{wn}) + mn > y_0 \mathbf{E}[e^{X_n}] \\ 0, & otherwise \end{cases} \tag{5}$$

represents (from the standpoint of date $n = 0$) the discounted expected value of the option's payoff if exercised specifically at future date $n \in \{0, \dots, N\}$. Combining Equations (4) and (5), we obtain for an arbitrage-free market

$$\mathbf{E}[u_n] \equiv \beta^n \mathbf{E}[(k(n) - Y_n)^+] = \begin{cases} \beta^n (k_0(1-e^{wn}) + mn - y_0), & if\ k_0(1-e^{wn}) + mn > y_0 \\ 0, & otherwise \end{cases}$$

$$= \beta^n \Pr(k_0(1-e^{wn}) + mn > y_0)(k_0(1-e^{wn}) + mn - y_0)^+$$

$$= \beta^n F_{X_n}\left(\ln\left(\frac{k_0(1-e^{wn})+mn}{y_0}\right)\right)(k_0(1-e^{wn}) + mn - y_0)^+ \text{ where } F_{X_n}(z) \equiv \Pr(X_n < z) \text{ for } z \in \Re$$

$$= \beta^n F_X\left(\frac{1}{n}\ln\left(\frac{k_0(1-e^{wn})+mn}{y_0}\right)\right)(k_0(1-e^{wn}) + mn - y_0)^+ \tag{6}$$

For a BG random variable X with $\alpha^+ = \alpha^- = 1$, we obtain from Küchler and Tappe (2008)

$$F_X(x) = \begin{cases} \frac{\lambda^+}{\lambda^+ + \lambda^-} e^{\lambda^- x}, & if\ x < 0 \\ \frac{\lambda^+}{\lambda^+ + \lambda^-} + \frac{\lambda^-}{\lambda^+ + \lambda^-}\left(1 - e^{-\lambda^+ x}\right), & if\ 0 \le x \le \infty \end{cases} \tag{7}$$

and since $\mathbf{E}[u_n] > 0 \Leftrightarrow \frac{1}{n}\ln\left(\frac{k_0(1-e^{wn})+mn}{y_0}\right) > 0$ it follows that Equations (6) and (7) yield

$$\mathbf{E}[u_n] = \beta^n \left(\frac{\lambda^+}{\lambda^+ + \lambda^-} + \frac{\lambda^-}{\lambda^+ + \lambda^-}\left(1 - e^{\left(-\frac{\lambda^+}{n}\ln\left(\frac{k_0(1-e^{wn})+mn}{y_0}\right)\right)}\right)\right)(k_0(1-e^{wn}) + mn - y_0)^+$$

$$= \beta^n \left(\frac{\lambda^+}{\lambda^+ + \lambda^-} + \frac{\lambda^-}{\lambda^+ + \lambda^-}\left(1 - \left(\frac{y_0}{k_0(1-e^{wn})+mn}\right)^{\frac{\lambda^+}{n}}\right)\right)(k_0(1-e^{wn}) + mn - y_0)^+ \tag{8}$$

Recall that π is some arbitrary exercise policy associated with a random exercise date $\tilde{n} \in \{0, \dots, N\}$. Hence, $\mathbf{E}[G^{\pi}(y_0)] = \mathbf{E}[u_{\tilde{n}}]$ and we can rewrite Problem (3) as

$$G^{\pi^*}(y_0) = max_{\mathbf{V}[u_{\tilde{n}}] \le \sigma^2_{max}} \mathbf{E}[u_{\tilde{n}}] = max_{\mathbf{V}[u_{\tilde{n}}] \le \sigma^2_{max}} \sum_{n=0}^{N} \Pr(\tilde{n} = n)\mathbf{E}[u_n] \tag{9}$$

Note that from the vantage point of date $= 0$, the value u_n is only strictly positive when y_0 lies strictly below the threshold $k_0(1-e^{wn}) + mn$. Hence, for each date $n \in \{0, \dots, N-1\}$ there should exist an optimal threshold price $b_n^* < \infty$ and $b_N^* = \infty$ such that it is optimal to exercise the option if the prevailing asset price $y_n < b_n^*$. That is, the optimal policy π^* must dictate that

$$a_n^{\pi^*}(y_n) = \begin{cases} 1, & if\ y_n < b_n^*\ \ for\ n \in \{0, \dots, N\} \\ 0, & otherwise \end{cases}$$

where the condition $b_N^* = \infty$ compels the option to be exercised by date N. The challenge therefore lies in computing the sequence of values $b_0^*, \dots, b_{N-1}^*$ associated with the optimal policy π^*. From the perspective of date $n = 0$, let $\Pr(a_n^\pi(Y_n) = 1)$ represent the probability of exercising the option *specifically* at future date $n \in \{0, \dots, N\}$. In the same vein, denote $p_n \equiv \Pr(\sum_{i=0}^n a_i^\pi(Y_i) = 1)$ as the probability of exercising before, or on date $n \in \{0, \dots, N\}$. By the independence of increments of Y_n, it follows that for $n \in \{1, \dots, N\}$, the probability $\Pr(a_n^\pi(Y_n) = 1)$ is connected to $p_n = \Pr(\sum_{i=0}^n a_i^\pi(Y_i) = 1)$ in the following manner.

$$\Pr(a_n^\pi(Y_n) = 1) = \Pr(\tilde{n} = n) = p_n - p_{n-1}$$
$$= \overbrace{\Pr(a_1^\pi(Y_1) = 1)}^{p_1} + \prod_{i=1}^{n-1}\left(1 - \Pr(a_i^\pi(Y_i) = 1)\right) \overbrace{\Pr\left(\sum_{i=0}^n a_i^\pi(Y_i) = 1\right)}^{p_n} \quad (10)$$

where $p_0 \leq p_1 \leq \cdots \leq p_N$ and the initial and terminal boundary conditions are

$p_0 \equiv 0$ because $k(0) = 0$ and $y_n < 0$ imply that $u_0 = 0$
$p_N \equiv 1$ because there is no reason not to exercise the option at its maturity date $n = N$

Remark 1. *As a result of Equation (10) and its boundary conditions, Problem (9)* is equivalently

$$G^{\pi^*}(y_0) = max_{\mathbf{V}[u_{\tilde{n}}] \leq \sigma_{max}^2} \sum_{n=1}^N (p_n - p_{n-1})\mathbf{E}[u_n] \ \ given\ p_0 = 0, p_0 = 1, p_0 \leq p_1 \leq \cdots \leq p_N \quad (11)$$

Furthermore,

$$\sum_{n=0}^N \Pr(a_n^\pi(Y_n) = 1) = p_1 + \sum_{n=2}^N \left(\prod_{i=1}^{n-1} \left(1 - \Pr(a_i^\pi(Y_i) = 1)\right) p_n \right) = \sum_{n=1}^N p_n - p_{n-1} = 1$$

and

$$p_n^* \equiv \Pr\left(\sum_{i=0}^n a_i^{\pi^*}(Y_i) = 1\right) = \Pr(Y_n < b_n^*) = \Pr(y_0 e^{X_n} < b_n^*)$$
$$= \Pr\left(X_n < \ln\left(\frac{b_n^*}{y_0}\right)\right) = F_{X_n}\left(\ln\left(\frac{b_n^*}{y_0}\right)\right) = F_X\left(\frac{1}{n}\ln\left(\frac{b_n^*}{y_0}\right)\right) \Leftrightarrow b_n^* = y_0 e^{nF_X^{-1}(p_n^*)} \quad (12)$$

where $F_X^{-1}(p) = \begin{cases} -\infty, & if\ p = 0 \\ \infty, & if\ p = 1 \end{cases}$ *and*

$$F_X^{-1}(p) = \begin{cases} \frac{1}{\lambda^-}\ln\left(p\left(1 + \frac{\lambda^-}{\lambda^+}\right)\right), & if\ 0 < p < \frac{\lambda^+}{\lambda^- + \lambda^+} \\ -\frac{1}{\lambda^+}\ln\left((1-p)\left(1 + \frac{\lambda^+}{\lambda^-}\right)\right), & if\ \frac{\lambda^+}{\lambda^- + \lambda^+} \leq p < 1 \end{cases} \quad (13)$$

is the quantile function of X *for any probability* $p \in [0,1]$.

From Equations (12) and (13), the threshold price for date $n \in \{0, \dots, N\}$ under the optimal exercise policy π^* is

$$b_n^* = \begin{cases} y_0 e^{n\left(\frac{1}{\lambda^-}\ln\left(p_n^*\left(1+\frac{\lambda^-}{\lambda^+}\right)\right)\right)} = y_0\left(p_n^*\left(1+\frac{\lambda^-}{\lambda^+}\right)\right)^{\frac{n}{\lambda^-}}, \; if\ 0 < p_n^* < \frac{\lambda^+}{\lambda^-+\lambda^+} \\ y_0 e^{n\left(-\frac{1}{\lambda^+}\ln\left((1-p_n^*)\left(1+\frac{\lambda^+}{\lambda^-}\right)\right)\right)} = y_0\left((1-p_n^*)\left(1+\frac{\lambda^+}{\lambda^-}\right)\right)^{-\frac{n}{\lambda^+}}, \; if\ \frac{\lambda^+}{\lambda^-+\lambda^+} \le p_n^* < 1 \end{cases} \tag{14}$$

and $b_n^* = \begin{cases} -\infty, \; if p_n^* = 0 \\ \infty, \; if\ p_n^* = 1 \end{cases}$. The nondecreasing sequence $p_0^* = 0, p_1^*, \dots, p_{N-1}^*, p_N^* = 1$ is obtained from the optimal solution vector for Problem (11). Lemma 2.3 of Cacoullos (1982) gives an upper bound for the variance of a function of a discrete random variable. Applying that lemma to $\mathbf{V}[u_{\tilde{n}}]$ we obtain

$$\mathbf{V}[u_{\tilde{n}}] \le \sum_{i=0}^{N-1}(\Delta\mathbf{E}[u_i])^2 \sum_{n=i+1}^{N} n(p_n - p_{n-1}) = \sum_{n=1}^{N} n(p_n - p_{n-1})\left(\sum_{i=0}^{n-1}(\Delta\mathbf{E}[u_i])^2\right)$$

where $\Delta\mathbf{E}[u_i] \equiv \mathbf{E}[u_{i+1}] - \mathbf{E}[u_i]$ and $\Delta\mathbf{E}[u_0] = \mathbf{E}[u_1] - \mathbf{E}[u_0] = \mathbf{E}[u_1]$ since $\mathbf{E}[u_0] = 0$. Hence, using Remark 1 we can express Problem (5) as the following linear optimization problem

$$G^{\pi^*}(y_0) = max_{\mathbf{p}, p_0=0, p_N=1} \sum_{n=1}^{N}(p_n - p_{n-1})\mathbf{E}[u_n] \tag{15}$$

subject to

$$\sum_{n=1}^{N} n(p_n - p_{n-1})\left(\sum_{i=0}^{n-1}(\Delta\mathbf{E}[u_i])^2\right) \le \sigma_{max}^2 \tag{16}$$

$$p_{n-1} \le p_n \text{ for all } n \in \{1, \dots, N\} \tag{17}$$

$$0 \le p_n \le 1 \text{ for all } n \in \{1, \dots, N\} \tag{18}$$

Proposition 2. *Suppose*

$$\hat{N} \equiv argmax_{n \in \{1,\dots,N\}}\, n\left(\sum_{i=0}^{n-1}(\Delta\mathbf{E}[u_i])^2\right) \le \sigma_{max}^2$$
$$N' \equiv argmax_{n \in \{1,\dots,N\}}\, \mathbf{E}[u_n]$$
$$N^* \equiv min\{\hat{N}, N'\}$$

Then a lower bound on the Bermudan put option's price is

$$\underline{G}^{\pi}(y_0) \equiv \sum_{n=1}^{N^*}\left(\frac{\frac{(N^*)^{2n}}{n!}}{\sum_{i=1}^{N^*}\frac{(N^*)^{2i}}{i!}}\right)\mathbf{E}[u_n] \tag{19}$$

Proof. See Appendix B.

The next section is an experimental investigation of the quality of our price bound.

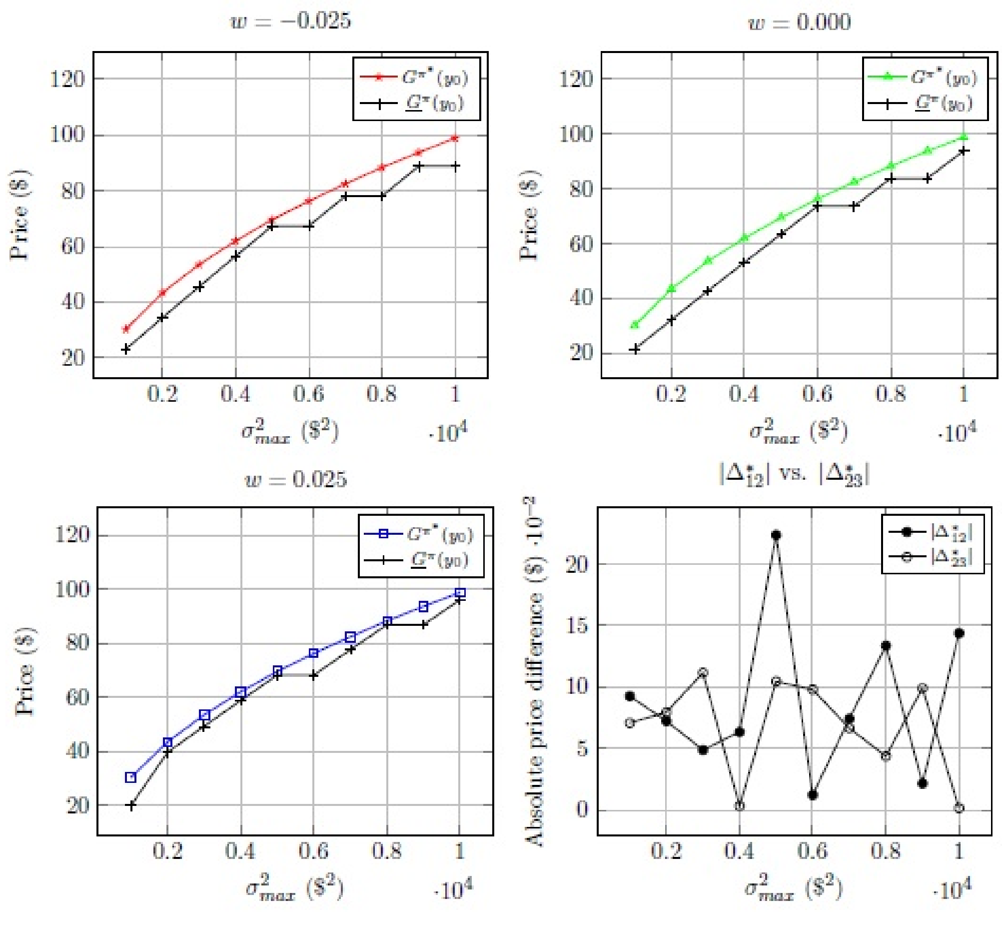
4 NUMERICAL STUDY

To measure how good our price bound is, we choose two different strike price functions, and then compare $G^{\pi^*}(y_0)$(from Problem (15)) with $\underline{G}^{\pi}(y_0)$ (from Equation (19)) for various values of w and σ^2_{max}(see Figure 1). We experiment on the following dataset: $m = \$10, \beta = 1, y_0 = \$5, k_0 = \$25, N = 24, \lambda^+ = 20, \lambda^- = 19,$ and define the following *price difference functions*

$$\Delta^*_{12} \equiv \left(G^{\pi^*}(y_0)\ for\ w = -0.025\right) - \left(G^{\pi^*}(y_0)\ for\ w = -0\right)$$

$$\Delta^*_{23} \equiv \left(G^{\pi^*}(y_0)\ for\ w = 0\right) - \left(G^{\pi^*}(y_0)\ for\ w = 0.025\right)$$

Figure 1: Effect of σ^2_{max} on $G^{\pi^*}(y_0), \underline{G}^{\pi}(y_0), |\Delta^*_{12}|,$ and $|\Delta^*_{23}|$



The experimentation suggests that the price lower bound ($\underline{G}^{\pi}(y_0)$) given by Equation (19) is relatively tight. In fact, in all test cases ($w = -0.025, w = 0$ and $w = 0.025$) the maximum percentage optimality gap is 26 percent. As expected, the option's price is a nondecreasing concave function of the payoff uncertainty budget, σ^2_{max}. This is consistent with the intuition that the expected payoff from exercising the option should increase as one's tolerance for payoff uncertainty increases. Alternatively stated, highly risk-averse prospective option buyers (i.e., those with low σ^2_{max}) should also be willing to pay little to own the option. Finally, the results suggest that the effect of σ^2_{max} on the option's absolute price difference functions (i.e., $|\Delta^*_{12}|$, and $|\Delta^*_{23}|$) is neither significant nor is indicative of a consistent relationship. This implies that the option's price dependence on σ^2_{max} is relatively insensitive to the sign of w which in turn plays an important role in governing the (deterministic) trajectory of the strike price over the time horizon.

5 CONCLUSION

In this paper, we considered the valuation of a Bermudan put option on an asset whose price fluctuations obey a simple version of a new model proposed by Küchler and Tappe (2008). We showed that the optimal exercise strategy dictates that the holder exercise the option as soon as the asset price drops below a certain threshold whose value (at every exercisable date) is efficiently computable from the vector solving a certain linear optimization problem. We characterized the option's price as the value associated to that solution vector, and heuristically established its lower bound. A numerical study confirmed the intuitive inverse relationship between risk-aversion and price, but also revealed that the aforementioned relationship is quite robust to changes in the nature of the strike price function.

We suggest two directions for further investigation. The obvious one would be to price the option for the general case when $\alpha^+ \neq 1$ and $\alpha^- \neq 1$. The other avenue would be to allow for situations in which the opportunity to exercise is modulated by some exogenous stochastic process. This would model the fact that sometimes option holders only have partial freedom to exercise their options (for instance, in situations where the right to exercise could be vetoed by some third party authority).

APPENDIX

A Proof of Proposition 1

Lemma 8.1 of Küchler and Tappe (2008) asserts that if unit time increments of X_t (under the measure **P**) follow a Γ (α^+, λ^+; α^-, λ^-) distribution, then assuming $\lambda^+ > 1$ and that $y_0 > 0$ is deterministic, the process $Y_t = y_0 e^{X_t}$ is a local **P**-martingale if and only if $\left(\frac{\lambda^+}{\lambda^+ - 1}\right)^{\alpha^+} = \left(\frac{\lambda^- + 1}{\lambda^-}\right)^{\alpha^-} \Rightarrow \left(\frac{\lambda^+}{\lambda^+ - 1}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + 1}\right)^{\alpha^-} = 1$. By Equation (1),

$$\mathbf{E}\left[e^{\theta X_n}\right] = \mathbf{E}\left[e^{\theta \sum_{i=1}^{n} X_i - X_{i-1}}\right] = \left(\left(\frac{\lambda^+}{\lambda^+ - \theta}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + \theta}\right)^{\alpha^-}\right)^n$$ for $\theta \in \Re$, $\lambda^+ > \theta$ and $n \in \{0, \dots, N\}$.

Therefore, for $\theta = 1$, $\mathbf{E}[e^{X_n}] = \mathbf{E}\left[e^{\sum_{i=1}^{n} X_i - X_{i-1}}\right] = \left(\left(\frac{\lambda^+}{\lambda^+ - \theta}\right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + \theta}\right)^{\alpha^-}\right)^n = 1^n = 1$ assuming $\lambda^+ > 1$.

B Proof of Proposition 2

By Constraint (16) $\sum_{n=1}^{N} n(p_n - p_{n-1})\left(\sum_{i=0}^{n-1} (\Delta \mathbf{E}[u_i])^2\right) \leq \sigma^2_{max} \Rightarrow \widehat{N}\left(\sum_{i=0}^{n-1} (\Delta \mathbf{E}[u_i])^2\right) \leq \sigma^2_{max}$ where $\widehat{N} \equiv argmax_{n \in \{1,\dots,N\}}\, n\left(\sum_{i=0}^{n-1} (\Delta \mathbf{E}[u_i])^2\right) \leq \sigma^2_{max}$. Now, for $n = 1$ to $n = N$ consider the sequence of probabilities $\{p_n - p_{n-1}\}_{n=1}^{\widehat{N}} = \left\{\frac{\frac{(\widehat{N})^{2n}}{n!}}{\sum_{i=1}^{N^*} \frac{(\widehat{N})^{2i}}{i!}}\right\}_{n=1}^{\widehat{N}}$ where $p_0 = 0$ and $p_{\widehat{N}} = p_{\widehat{N}+1} = \cdots = p_N = 1$.

Since $\sum_{n=1}^{N} p_n - p_{n-1} = \sum_{n=1}^{N} \frac{\frac{(\widehat{N})^{2n}}{n!}}{\sum_{i=1}^{N^*} \frac{(\widehat{N})^{2i}}{i!}} = 1$, it follows that $p_0 = 0$ and $p_{\widehat{N}} = p_{\widehat{N}+1} = \cdots = p_N = 1$, the sequence $\{p_n - p_{n-1}\}_{n=1}^{\widehat{N}} = \left\{ \frac{\frac{(\widehat{N})^{2n}}{n!}}{\sum_{i=1}^{N^*} \frac{(\widehat{N})^{2i}}{i!}} \right\}_{n=1}^{\widehat{N}}$ is a feasible solution for Problem (15) yielding $\sum_{n=1}^{\widehat{N}} \frac{\frac{(\widehat{N})^{2n}}{n!}}{\sum_{i=1}^{N^*} \frac{(\widehat{N})^{2i}}{i!}} \mathbf{E}[u_n]$ as a lower bound on the option's price. However, by defining $N' \equiv argmax_{n \in \{1,\ldots,N\}} \mathbf{E}[u_n]$, $N^* \equiv min\{\widehat{N}, N'\}$ and noticing that $\mathbf{E}[u_n]$ is either concave or monotonically increasing in n, we can obtain an improved lower bound if we replace $\widehat{N}$ by N^*. Hence, $\underline{G}^{\pi}(y_0) \equiv \sum_{n=1}^{N^*} \left(\frac{\frac{(N^*)^{2n}}{n!}}{\sum_{i=1}^{N^*} \frac{(N^*)^{2i}}{i!}} \right) \mathbf{E}[u_n] \leq G^{\pi^*}(y_0)$.

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