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Multiple Criteria Portfolio Choice with Variance Decomposition onto Half Spaces

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ABSTRACT

In this paper we propose a multiple criteria framework (MCF) for generalizing the traditional two-moment CAPM. It includes the Sharpe-Lintner CAPM as a special case. The model decomposes effective portfolio variance into two parts, an undesirable total variance part and a positive variability part. Minimization is taken with respect not to total variance as in the Sharpe-Lintner CAPM, but to the total variance less the desirable positive variability. Positive covariability reduces risk premium, and has a similar effect compared to positive skewness or upper partial moment. The empirical results in our MCF model indicate good performance in the model.

KEYWORDS: Multi-Criteria Optimization, Portfolio Selection, Variance Decomposition

INTRODUCTION

The pioneering work of (Markowitz, 1952) in developing the mean-variance efficient portfolio frontier has remained a dominant pillar in both practices of investment as well as in financial capital asset pricing theory. Literally thousands of published papers, including at least a couple of hundreds in the few leading academic finance journals, are built upon the premise of the mean-variance portfolio optimization. Although there had been a considerable number of extensions and alternatives, single-period mean-variance two-parameter (or two-moment) optimization and its equilibrium capital asset pricing model (CAPM) by (Sharpe, 1964), (Lintner, 1965), and the zero-beta version by (Black, 1972) continue to be popular in ongoing academic textbooks and is taught in almost every basic investment class spanning undergraduate curriculum to MBA and executive programs. The analytical tractability of the two-moment portfolio frontier and the corresponding equilibrium asset pricing model is a prized finding of classical finance research. Mathematically, the geometry of quadratic programming in mean-variance portfolio optimization leads to a unique analytical solution of a linear relationship in expected return and weighted average of all expected returns that is the expected market return.

If we use the classical von Neumann-Morgenstern (VM) expected utility framework, mean-variance optimization is theoretically sound under either quadratic utility, or else multivariate elliptical return distributions. (Chamberlain, 1983) showed the elliptical distribution class produced mean-variance efficiency under only the condition of increasing and concave utility functions. The elliptical distributional family includes the multivariate normal distribution, mixture normal, multivariate t, multivariate stable, Kotz, and Pearson II distributions with the property of linear conditional expectations. (Cass and Stiglitz, 1970) showed that if and only if agents in the economy have hyperbolic absolute risk aversion (HARA) utility functions, where the utility functions are increasing and concave in end-of-period wealth, then two-fund separation can be shown to exist in markets where a risk-free asset exists. The HARA function that is considerably general in the context of Von-Neumann Morgenstern expected utility theory includes the linear, the quadratic, the negative exponential (CARA), and power (CRRA) functions. Two-fund separation is certainly consistent with, but does not necessarily lead to the Sharpe-Lintner CAPM, although it does so if returns are expressible in a linear regression form on market return with an uncorrelated residual noise.

However, as is well-known, mean-variance optimization is rather restrictive. In fact, it is more restrictive than is necessary. In this paper we show that it can be suitably extended to a multiple criteria framework (MCF) that conforms with expected VM utility maximization as well as the aggregation of individual preferences of the HARA form such as shown in (Rubinstein, 1974) under general incomplete market equilibrium condition. The MCF model in this paper consists of the traditional undesirable total variance and a desirable part comprising positive variability. Minimization is taken with respect not to total variance as in the Sharpe-Lintner CAPM, but to the total variance less the desirable positive criteria, which we term as multiple criteria variance (MCV). This generalized model embodies all the advantages and prized classical results of easy computation of optimal portfolio weights and thus establishment of uniqueness and existence of a maximum expected utility and also properties of linear risk sharing and Pareto efficiency. This original idea of positive variability and of the associated positive covariability as a risk measure is such that positive covariability reduces risk premium, and has a similar effect compared to positive skewness in (Kraus-Litzenberger, 1976) three-moment CAPM or in the lower partial moment models of (Hogan and Warren, 1974), and of (Bawa and Lindenberg, 1977). As in the Sharpe-Lintner CAPM, our MCF model is consistent with the assumption of HARA utility or the assumption of elliptical return distributions. The latter accommodates return distributions that are multivariate normal, and also various other return distributions with skewness.

We shall defer a more lengthy discussion of the motivations of the idea of positive variability and positive covariability to the next section. To get into gear, we state that the crux of our thesis of the non-trivial extension of the classical two-moment CAPM. Let the continuously compounded return rate of a stock i at time $t+1$ be $r_{i,t+1} = \ln \frac{p_{i,t+1}}{p_{i,t}}$ where $p_{i,t}$ is the corresponding stock price at time t. We shall require only the assumptions that $r_{i,t+1}$ has a probability measure defined over $(-\infty, +\infty)$, and that its first two moments exist, i.e. $E(r_{i,t+1}) < \infty$ and $E(r_{i,t+1}^2) < \infty$. Now $r_{i,t+1} = r_{i,t+1}^+ + r_{i,t+1}^-$, where $r_{i,t+1}^+ = \max(r_{i,t+1}, 0)$ and $r_{i,t+1}^- = \min(r_{i,t+1}, 0)$.

Basically we require the use of an indirect or expected utility function $V(\mu_p, \sigma_p^2, \sigma_p^{+2})$ where the new entity σ_p^{+2} proxies for some form of skewness or sensitivity of the utility function beyond just mean and variance. For N risky stocks in the economy, $\mu_p = \sum_{i=1}^N w_i E(r_{i,t})$, $\sigma_p^2 = \text{var}\left(\sum_{i=1}^N w_i r_{i,t}\right)$, and $\sigma_p^{+2} = \text{var}\left(\sum_{i=1}^N w_i r_{i,t}^+\right)$, where $E(\cdot)$ is the expectation operator, w_i is the portfolio weight of the i^{th} stock and $\sum_{i=1}^N w_i = 1$. As in the traditional Sharpe two-moment CAPM utility function $V(\mu_p, \sigma_p^2)$, V is increasing in μ_p and decreasing in σ_p^2 with these quantities representing portfolio mean return and portfolio return variance respectively. Additionally, our V is increasing in the new entity σ_p^{+2}. We shall assume the existence of a zero net supply of riskless asset and use the U.S. short-term Treasury yield as a measure of its return rate $r_{f,t}$. This could be seen as facilitating the existence of monetary fund separation, and in expressions involving excess return rate, it has the advantage of canceling out the inflationary components on both nominal stock returns and riskless returns, thus giving rise to a real return rate that is more in line with the spirit of the one-period model without consideration of consumption good inflation over that period. Risky assets consist of the stocks in the economy that are in net positive supply.

With the decomposition of the return rate, we have

$$E_t\left(r_{i,t+1}\right)=E_t\left(r^+_{i,t+1}\right)+E_t\left(r^-_{i,t+1}\right), \tag{1}$$

where subscript t to the expectation operator denotes the expectation is taken with respect to information Φ_t available at time t. The conditional expectation could alternatively be written as $E\left(\cdot|\Phi_t\right)$. Therefore, $r_{i,t+1}-E_t\left(r_{i,t+1}\right)=\left(r^+_{i,t+1}-E_t\left(r^+_{i,t+1}\right)\right)+\left(r^-_{i,t+1}-E_t\left(r^-_{i,t+1}\right)\right)$. To simplify notations, we let the conditional means $E_t\left(r_{i,t+1}\right)=\mu_{i,t}$, $E_t\left(r^+_{i,t+1}\right)=\mu^+_{i,t}$, and $E_t\left(r^-_{i,t+1}\right)=\mu^-_{i,t}$. Thus,

$$E_t\left(r_{i,t+1}-\mu_{i,t}\right)^2=E_t\left(r^+_{i,t+1}-\mu^+_{i,t}\right)^2+2\;E_t\left(r^+_{i,t+1}-\mu^+_{i,t}\right)\left(r^-_{i,t+1}-\mu^-_{i,t}\right)+E_t\left(r^-_{i,t+1}-\mu^-_{i,t}\right)^2. \tag{2}$$

We can re-write Eq.(2) as

$$\text{var}_t\left(r_{i,t+1}\right)=E_t\left(r^+_{i,t+1}\right)^2-\left(\mu^+_{i,t}\right)^2-2\mu^+_{i,t}\mu^-_{i,t}+E_t\left(r^-_{i,t+1}\right)^2-\left(\mu^-_{i,t}\right)^2, \tag{3}$$

where $\text{var}_t(\cdot)$ denotes conditional variance. It is noted that $E_t\left(r^+_{i,t+1}r^-_{i,t+1}\right)=0$. The decomposition of the random variable into projections on half-spaces introduces some simple but unique functions such as the terms on the right-hand side of Eq.(3). These terms involve at most the second moments of the corresponding random variables.

For the return rate $r_{i,t+1}$, the third central moment, if it exists, can be expressed as:

$$\begin{aligned}
E_t[r_{i,t+1}-\mu_{i,t}]^3 &= E_t[r^+_{i,t+1}-\mu^+_{i,t}]^3+E_t[r^-_{i,t+1}-\mu^-_{i,t}]^3 \\
&\quad +3E_t[(r^+_{i,t+1})^2r^-_{i,t+1}]-3\mu^-_{i,t}E_t[(r^+_{i,t+1})^2]+6(\mu^+_{i,t})^2\mu^-_{i,t} \\
&\quad +3E_t[r^+_{i,t+1}(r^-_{i,t+1})^2]-3\mu^+_{i,t}E_t[(r^-_{i,t+1})^2]+6\mu^+_{i,t}(\mu^-_{i,t})^2 \\
&= E_t[r^+_{i,t+1}-\mu^+_{i,t}]^3+E_t[r^-_{i,t+1}-\mu^-_{i,t}]^3 \\
&\quad -3\mu^-_{i,t}E_t[(r^+_{i,t+1})^2]-3\mu^+_{i,t}E_t[(r^-_{i,t+1})^2]+6\mu^+_{i,t}\mu^-_{i,t}\mu_{i,t}.
\end{aligned} \tag{4}$$

From the above Eq.(4), ceteris paribus, the left-hand side third central moment with corresponding skewness increases if $E[(r^+_{i,t+1})]^2$ increases and $E[(r^-_{i,t+1})]^2$ decreases. The left-hand side skewness decreases if $E[(r^+_{i,t+1})]^2$ decreases and $E[(r^-_{i,t+1})]^2$ increases. Obviously this is just an intuition on the possible effect and direction of effect of variations in $r^+_{i,t+1}$ and $r^-_{i,t+1}$ on the third central moment and skewness. Exact effects can only be established given the distribution or its analytical moments.

Variance Decomposition in Portfolio Optimization

We next characterize the use of the decomposed terms more generally in the context of a multivariate distribution. Specifically, let the portfolio of N stock returns be denoted by $r_{i,t+1}=r^+_{i,t+1}+r^-_{i,t+1}$ for $i=1,2,\ldots,N$. For the following, we shall drop the subscript time notation. Though this is a one-period model, when we perform empirical cross-sectional time series testing, we would assume that the model holds for every t.

The covariance matrix of the random return vector is:

$$V_{N\times N}=E\left[\begin{pmatrix}(r_1^+-\mu_1^+)+(r_1^--\mu_1^-)\\(r_2^+-\mu_2^+)+(r_2^--\mu_2^-)\\\vdots\\\vdots\\(r_N^+-\mu_N^+)+(r_N^--\mu_N^-)\end{pmatrix}\Big((r_1^+-\mu_1^+)+(r_1^--\mu_1^-)\ \dots(r_N^+-\mu_N^+)+(r_N^--\mu_N^-)\Big)\right],$$

where the ij^{th} term of the $N\times N$ matrix V is: cov$(r_i^+,r_j^+)+$cov$(r_i^+,r_j^-)+$cov$(r_i^-,r_j^+)+$cov(r_i^-,r_j^-).

The $N\times N$ covariance matrix V can be expressed as:

$$V_{N\times N}=V_{N\times N}^{++}+V_{N\times N}^{+-}+V_{N\times N}^{-+}+V_{N\times N}^{--}, \tag{5}$$

where

$$V_{N\times N}^{++}=\begin{pmatrix}\text{var}(r_1^+) & \text{cov}(r_1^+,r_2^+) & \dots & \text{cov}(r_1^+,r_N^+)\\ \text{cov}(r_2^+,r_1^+) & \text{var}(r_2^+) & \dots & \text{cov}(r_2^+,r_N^+)\\ \vdots & \vdots & \ddots & \vdots\\ \text{cov}(r_N^+,r_1^+) & \text{cov}(r_N^+,r_2^+) & \dots & \text{var}(r_N^+)\end{pmatrix},$$

$$V_{N\times N}^{+-}=\left(V_{N\times N}^{-+}\right)'=\begin{pmatrix}\text{cov}(r_1^+,r_1^-) & \text{cov}(r_1^+,r_2^-) & \dots & \text{cov}(r_1^+,r_N^-)\\ \text{cov}(r_2^+,r_1^-) & \text{cov}(r_2^+,r_2^-) & \dots & \text{cov}(r_2^+,r_N^-)\\ \vdots & \vdots & \ddots & \vdots\\ \text{cov}(r_N^+,r_1^-) & \text{cov}(r_N^+,r_2^-) & \dots & \text{cov}(r_N^+,r_N^-)\end{pmatrix},$$

and

$$V_{N\times N}^{--}=\begin{pmatrix}\text{var}(r_1^-) & \text{cov}(r_1^-,r_2^-) & \dots & \text{cov}(r_1^-,r_N^-)\\ \text{cov}(r_2^-,r_1^-) & \text{var}(r_2^-) & \dots & \text{cov}(r_2^-,r_N^-)\\ \vdots & \vdots & \ddots & \vdots\\ \text{cov}(r_N^-,r_1^-) & \text{cov}(r_N^-,r_2^-) & \dots & \text{var}(r_N^-)\end{pmatrix}.$$

The above variance decomposition provides for interesting analytical possibilities as well as meaningful financial economics. Variance decomposition in asset pricing may be a relatively new idea, though return decomposition has been studied elsewhere in other fields, including areas of optimal risk management as in (Natarajan, Pachamanova, and Sim, 2008), and (Goh, Lim, Sim, and Zhang, 2012). We can construct a long position on stock i as a portfolio of long one unit of call on stock i at day t with strike price $p_{i,t}$ and payoff the next maturity day of $\max(p_{i,t+1}-p_{i,t},0)$, short one unit of put on stock i at day t with strike $p_{i,t}$ and payoff next day of $\min(p_{i,t+1}-p_{i,t},0)$, and lending at day t of amount $p_{i,t}/(1+r_f^*)$ for a day, where r_f^* is the daily effective riskless rate.

While $w'Vw=\sigma_p^2$, w being the $N\times 1$ vector of portfolio weights, is clearly the usual portfolio variance, the term $w'V^{++}w=\sigma_p^{+2}$ is the variance of a portfolio of long call returns, while the term $w'V^{--}w=\sigma_p^{-2}$ is the variance of another portfolio of short put returns (or losses). Just as underlying stock volatility increases long call position values and returns (and similarly increases variance of the future call price - using for example the Black-Scholes log-diffusion model), a volatility increase in σ_p^{+2} per se can be seen to be valuable and desirable to the portfolio investor. However, on the contrary, stock volatility also increases put position values and thus losses since these are short positions. In the latter, such a volatility increase in σ_p^{-2} or σ_p^2 in general can be seen to be undesirable to the portfolio investor. Using option theory, we therefore attain an intuitive explanation of why it is relevant and important to consider the presence of σ_p^{+2} and σ_p^{-2}. Since undesirable σ_p^{-2} is already subsumed in undesirable total variance σ_p^2, only valuable σ_p^{+2} needs to be separated out as an additional explanatory variable characterizing the more general mean-variance-skewness preference $V(\mu_p, \sigma_p^2, \sigma_p^{+2})$. This produces $\sigma_p^{+2})$ as the added criterion in the MCF.

Now, $w'Vw = w'\left(V^{++} + 2V^{+-} + V^{--}\right)w = w'V^{++}w + w'\left(2V^{+-} + V^{--}\right)w$. On the right-hand side, the first term $w'V^{++}w$ is the variance of a portfolio of positive components of returns, and thus improves utility. The remaining terms represent the part of a portfolio variance that either decreases utility, or has no clear benefit to utility. The traditional Markowitz portfolio optimization theory does not distinguish the positive effect of $w'V^{++}w$ on utility, but incorporates the term in the total portfolio variance as a negative impact on utility. It is equivalent to specifying a utility function that is not refined to account for the positive contribution to utility of variations in $r_{i,t+1}^{+}$. To account for this, the idea is not to minimize total variance of portfolio return, but to minimize the total variance less the positive variability, or the MCV.

In section 2 we explain the MCF objective function and provide the solution to an extended two-moment asset pricing model. Section 3 provides an empirical test of our model using the method of (Black, Jensen, and Scholes, 1972) and (Fama and McBeth, 1973). The empirical results are compared with the Kraus-Litzenberger three-moment asset pricing model and also the lower partial moment model. Section 4 contains the summary and conclusions.

MULTI-CRITERIA VARIANCE

To improve on the mean-variance portfolio optimization model without the need to approximating utility function by Taylor series expansion to incorporate skewness, we can capture skewness effect, without necessarily providing an explicit definition of skewness or assuming existence of the third moment, by minimizing risk as represented by the MCV, i.e. total variance less positive variability.

$$\min_{w} \frac{1}{2} w'\left(V - \alpha V^{++}\right)w + \lambda(k - w'\mu - [1 - w'L]r_f) \tag{6}$$

where k is the minimum required rate of return of the overall portfolio including investing in the riskfree asset, L is an $N \times 1$ vector of ones, and $\alpha \geq 0$ is a parameter to be discussed in more detail. μ is a $N \times 1$ vector of the expected returns of the N stocks.

It can be readily shown that $V - \alpha V^{++}$ is symmetric positive definite for α^{-1} larger than all characteristic roots of $V - \alpha V^{++}$. α is chosen suitably. We now exposit the popular and long-used mean-variance framework for portfolio optimization and asset pricing. In typical financial economics, asset pricing is derived via equilibrium conditions imposed over and above the conditions leading to portfolio optimization. Equilibrium conditions would include aggregation across individual investors in the market to produce zero excess demand. We now solve analytically the portfolio optimization problem in (6) and also impose market equilibrium conditions.

The first-order optimality conditions for (6) are:

$$(V - \alpha V^{++})w = \lambda(\mu - r_f L)\,, \tag{7}$$

and

$$k = w'\mu + [1 - w'L]r_f\,. \tag{8}$$

Eq.(8) can also be written as $k = w'(\mu - r_f L) + r_f$.

From Eqs. (7) and (8), we obtain

$$\begin{aligned} w'(V - \alpha V^{++})w &= \lambda w'(\mu - r_f L) \text{ , or} \\ w'Vw - \alpha w'V^{++}w &= \lambda\left(w'\mu - r_f\right) \text{ , or} \\ \sigma_p^2 - \alpha {\sigma_p^+}^2 &= \lambda(k - r_f). \end{aligned}$$

In equilibrium when risky assets are in net positive supplies, when there is a representative agent or investor, then k is the return on a market portfolio of the risky securities. Denote $k = \mu_M$. In this case, $w'Vw = \sigma_M^2$ is the market portfolio variance, and $w'V^{++}w = {\sigma_M^+}^2$ is the market portfolio positive variability. $\sigma_M^2 - \alpha{\sigma_M^+}^2$ is the market MCF variance with the α factor. Thus, $\lambda = \frac{\sigma_M^2 - \alpha{\sigma_M^+}^2}{\mu_M - r_f}$ is a measure of the market risk (as measured by MCF variance) to excess return tradeoff, or cost in terms of risk per unit of return in excess of the risk-free rate. This is the shadow price of the return constraint in the risk minimization program of (6). The more risk-averse is the representative agent, the smaller is this value of λ in market equilibrium. The inverse of this optimal λ is similar to the idea of the Sharpe ratio except that the risk is measured by subvariance than variance. In the special case when $\alpha = 0$, we obtain the usual Sharpe-Lintner CAPM.

The solution to Eqs. (7) and (8) for optimal portfolio weight vector is

$$w = \lambda\left(V - \alpha V^{++}\right)^{-1}(\mu - r_f L) = \frac{\sigma_M^2 - \alpha{\sigma_M^+}^2}{\mu_M - r_f}\left(V - \alpha V^{++}\right)^{-1}(\mu - r_f L) \ .$$

In fact in our MC framework, as in mean-variance optimization, this solution is unique and easily computed. Such a solution is a necessary condition for the asset pricing relationship that follows in the determination of equilibrium risk-adjusted expected returns on securities in the market. Some asset pricing models do not show the existence and uniqueness of the portfolio weights. The optimality of the latter are simply assumed in the first order conditions. Such an implicit assumption may not be correct.

The above minimization program can be solved in a more detailed way by involving the utility function. Let the expected utility function be $U(\mu_p, \sigma_p^2, {\sigma_p^+}^2)$ where $a = \frac{\partial U}{\partial \mu_p} > 0$, $b = \frac{\partial U}{\partial \sigma_p^2} < 0$, $c = \frac{\partial U}{\partial {\sigma_p^+}^2} > 0$, $\sigma_p^2 = w'Vw$, ${\sigma_p^+}^2 = w'V^{++}w$, and w is the optimal portfolio $N \times 1$ vector of weights with $w\prime L = 1$.

Taking the total derivative of U, for each individual $j \in [1, M]$,

$$a_j \frac{d\mu_p}{dw} + b_j \frac{d\sigma_p^2}{dw} + c_j \frac{d{\sigma_p^+}^2}{dw} = 0 \ .$$

Multiplying by W^j, the wealth of investor j, we obtain

$$-\frac{a_j}{2b_j} W^j(\mu - r_f L) = Vw_j W^j + \frac{c_j}{b_j} V^{++} w_j W^j \ .$$

Aggregating across M investors in the market economy, and dividing by M,

$$\begin{aligned} &[\mu - r_f L]\sum_{j=1}^{M}\left(-\frac{a_j}{2b_j}\right)W^j/M = V\sum_{j=1}^{M} w_j W^j/M + V^{++}\sum_{j=1}^{M}\frac{c_j}{b_j} w_j W^j/M \ , \quad \text{or} \\ &T^M[\mu - r_f L] = Vw_M - \alpha V^{++} w_M \qquad (9) \end{aligned}$$

where $T^M = \sum_{j=1}^{M} T^j/M$, $T^j = -(a_j W^j)/(2b_j)$ being the individual risk tolerance, w_M is the aggregated market portfolio weight vector that is optimally selected by the representative agent, and $c_j/b_j = -\alpha$ where $\alpha > 0$ is assumed to be a constant. T^M is the aggregated market risk tolerance or the risk tolerance of the representative agent.

We note that $\alpha = -\frac{c_j}{b_j} = -\frac{\partial U}{\partial \sigma_p^{+2}} / \frac{\partial U}{\partial \sigma_p^2} > 0$. Along a constant indifference curve of the utility function, $\alpha = \frac{\partial \sigma_p^2}{\partial \sigma_p^{+2}}$ which is an equilibrium market cost of increase in portfolio return variance per unit increase in positive variability. This can also be interpreted as the reward or value of per unit positive variability. The more valuable is positive variability, the higher is the positive value of α.

From Eq. (9),

$$T^M w_M'[\mu - r_f L] = w_M' V w_M - \alpha w_M' V^{++} w_M. \tag{10}$$

Or,

$$T^M[\mu_M - r_f] = \sigma_M^2 - \alpha {\sigma_M^+}^2,$$

where μ_M is the market return, σ_M^2 is the market return variance, and ${\sigma_M^+}^2$ is the variance of $\sum_{i=1}^{N} w_i r_i^+$.

Substituting for T^M in Eq. (9), we obtain the generalized model:

$$\mu - r_f L = \frac{V w_M - \alpha V^{++} w_M}{\sigma_M^2 - \alpha {\sigma_M^+}^2} \left(\mu_M - r_f\right). \tag{11}$$

Hence, the MCF model of asset pricing shows that for each security i, its equilibrium expected return is given by

$$E(r_i) - r_f = \frac{\text{cov}(r_i, r_M) - \alpha \text{cov}(r_i^+, \sum_{i=1}^{N} w_i r_i^+)}{\sigma_M^2 - \alpha {\sigma_M^+}^2} \left(\mu_M - r_f\right). \tag{12}$$

The result in Eq. (12) clearly has the two-fund separation property as in the Sharpe-Lintner CAPM or the mean-lower partial moment (LM) asset pricing model of Hogan and Warren (1974) and Bawa and Lindenberg (1977). Eq. (12) also appears to be rather similar in form to the LM model:

$$E(r_i) - r_f = \frac{E\left((r_i - r_f)(r_M - r_f)^-\right)}{E\left([r_m - r_f]^-\right)^2} \left(\mu_M - r_f\right);,$$

where $[r_m - r_f]^- = \left(\sum_{i=1}^{N} [w_i r_i] - r_f\right)^-$. Both MCF and LM models have two-fund separation, though the slopes on the risk premium of $(\mu_M - r_f)$ are different, and are also different from that in the Sharpe-Lintner mean-variance model. The LM model converges to the Sharpe-Lintner model when the underlying return distribution becomes multivariate normal and necessarily symmetrical. Likewise, the KL model converges to the Sharpe-Lintner model when skewness is zero. In MCF model, the special case of Sharpe-Lintner model obtains when the value of positive variability, α, becomes zero, while the return distribution converges to elliptical form that is not necessarily symmetrical; the value of α is certainly an empirical property. Both MCF and LM models would appear to imply HARA utility for market equilibrium, though MCF allows for the aggregation property of a representative agent while LM does not appear to provide for aggregation, and needs the usual assumption of homogeneous agents.

Our MCF asset pricing model in Eq. (12) indicates that if $\alpha = 0$, then we have the special case of the Sharpe-Lintner two-moment CAPM. When $\alpha > 0$, positive covariability, or covariance of r_i^+

with $\sum_{i=1}^{N} w_i r_i^+$ that leads to higher portfolio positive variability is a desirable attribute of the stock that lowers its equilibrium required rate of return. Lower positive covariability does not lead to as much reduction in expected return, and thus implies a higher risk premium relative to the case with higher positive covariability. α is thus the reward for positive covariability. We suggest that when the market skewness is more negative, positive covariability becomes more of a rarer commodity and thus fetches a higher reward in terms of the outcome of a lower market risk premium. Thus we may see a higher α when the market is seeing larger losses than gains. The concept of positive covariability is original in the literature, and allows for linear aggregation as in beta and co-skewness measures.

EMPIRICAL RESULTS

In this section, we shall empirically test the MCF model together with the KL model and the LM model using the cross-sectional approach of (Black, Jensen, and Scholes, 1972), (Fama and MacBeth, 1973), (Kraus and Litzenberger, 1976), and (Fama and French, 1992). Generally, a two-step procedure is used in such a cross-sectional approach to asset pricing test. Other approaches to asset pricing test include time series cross-sectional tests such as (Gibbons, 1982) where assuming unconditional stationarity of returns series, and assuming an efficient market return that is observable, more powerful test statistic can be developed invoking the cross-sectional restrictions on the coefficients. Another approach is based on conditional stationarity, but imposes informational structure on the changing betas or other risk attributes, such as in (Campbell and Siddique, 2000). First, stock attributes or risk factors, for example, its beta and co-skewness measure (gamma), are estimated by ordinary least squares (OLS) using monthly time series in the immediate past number of years. Stocks are formed into portfolios where each portfolio contains stocks with estimated attributes that are similar within defined percentiles. By taking equal-weighted (or sometimes value-weighted) portfolios of the stocks, the attributes of the portfolio can be estimated with much higher precision to the extent beta and gamma are the appropriate information to condition on.

(Kraus and Litzenberger, 1976) provided a detailed justification of this procedure, citing significant reduction of estimation errors or “errors-in-variable” bias, and also improved stability of the regressor estimates. (Black, Jensen, and Scholes, 1972) noted a possible small correlation between grouped regressor estimates and the direction of systematic measurement errors. (Malinvaud, 1966) had suggested the use of instrumental variables that are independent of the direction of systematic measurement errors, but correlated with the true value of the regressor variables.

In what follows, when we perform grouping procedures for finding regressors of the generalized two-moment model, we de facto use the beta and co-skewness measure in KL model as instrumental variables since co-skewness or gamma is also a good instrument or proxy for skewness measure as in MCF’s positive covariability, and LM’s upper partial moment. Bossaerts (2002) noted that the procedure amounted to estimated regressors conditional on past betas and past gammas, and that use of the relevant information set for conditioning is important. Appropriate conditioning may improve the power of the asset pricing test, while conditioning based on wrong information set may lead to easier rejection of the asset pricing model.

Specifically, we follow the procedure of Kraus and Litzenberger in sorting by betas and gammas. For each past 10 years starting with January 1926 to December 1935, and rolling forward one-year at a time, the stocks are sorted into 25 portfolios using their estimated betas and co-skewness measures (gammas) from the 10 years of monthly stock return data. Stocks with the lowest betas are sorted into the first quintile. Then the stocks within this quintile are again sorted by co-skewness into quintiles. Stocks with betas in the second quintile are sorted by co-skewness into quintiles, and so on. Thus, there are 5 by 5 or 25 portfolios.

In the second step, the time series of stock returns within each portfolio are then used to find the ex-post risk measures corresponding to each of the three models. A set of the risk measures is estimated for each month. To avoid correlation bias with the residual error term of the cross-sectional regression in each month, the risk measures are estimated using the entire ex-post time series but excluding the return data for that particular month in the estimation.

For MCF model, the cross-sectional explanatory variables for each month are the sample estimates of $X_{1i} = \frac{\text{cov}(r_i, r_M) - \alpha \text{cov}(r_i^+, \sum_{i=1}^N w_i r_i^+)}{(\sigma_M^2 - \alpha {\sigma_M^+}^2)}$. One important point to note in the empirical testing here is that the MCF model requires the measure of positive covariability cov$(r_i^+, \sum_{i=1}^N w_i r_i^+)$ which cannot be obtained from reported market return data. Therefore we use the entire sample of stock returns and the reported outstanding shares for each stock to construct stock weights and individual components of positive variability for each stock, in order to construct the measure of positive covariability in MCF. To be consistent, the market return used in this paper should also be the sum total of the value-weighted return of all stocks in our sample. For consistency check, we note there is no significant difference from reported value-weighted market index.

For KL, the explanatory variables are the sample estimates of $Y_{1i} = \text{cov}(r_i, r_M)/\sigma_M^2$ and $Y_{2i} = E((r_i - \mu_i)(r_M - \mu_M)^2)/[E(r_M - \mu_M)]^{2/3}$. The latter explanatory variable is gamma times the cube root of the third central moment of the market return. It allows for the interpretation of the corresponding coefficient, the marginal rate of substitution between expected wealth and wealth skewness, to be negative along a constant indifference curve. For LM, the cross-sectional explanatory variables are the sample estimates of $Z_{i1} = E((r_i - r_f)(r_M - r_f)^-)/E([r_M - r_f]^-)^2$. All three models can be written as the following cross-sectional regressions across portfolios $i = 1, 2, \ldots, 25$, for each month of January 1936 to December 2011.

$$\begin{aligned} MCF &: \quad r_i - r_f = a_0 + a_1 X_{1i}(\alpha) + e_i \ , \\ KL &: \quad r_i - r_f = b_0 + b_1 Y_{1i} + b_2 Y_{2i} + e_i \ , \\ LM &: \quad r_i - r_f = c_0 + c_1 Z_{1i} + e_i \ , \end{aligned}$$

where e_i is a white noise. We employ all stock return rate data from CRSP over the period January 1926 to December 2011. To be consistent with the use of monthly return computation, we choose to use the reported monthly risk-free rate in (Fama and French, 1992) instead of the 3-month rate in CRSP. One point to note for comparison with results from earlier studies such as in (Kraus and Litzenberger, 1976) is that though the results are not significantly different, there are small differences due to the use of different data construction from current CRSP data.

For each month starting from January 1936, cross-sectional regression across all formed portfolios is performed by OLS regression of portfolio returns on the portfolios' estimated attributes to find the risk premia $\hat{a}_1$, $\hat{b}_1$, $\hat{b}_2$, and $\hat{c}_1$ respectively. For MCF, there is the additional estimate of $\hat{\alpha}$ by performing nonlinear least squares estimation for this case.

Suppose there are T months in which formed portfolios are used for cross-sectional regressions. Then there are T number of estimated risk premia for each month. These estimates or the cross-sectional regression estimated coefficients are then averaged over T. The t-statistic of the deviation of the cross-sectional estimate from the time series average is also computed. The averaged adjusted R^2's are also reported.

Table 1: Table shows average beta and average co-skewness (or systematic skewness) measure of stocks sorted into 25 portfolios. The stocks are sorted using their estimated betas and co-skewness measures (gammas) from the previous 10 years of monthly stock return data. Each year forward since 1936, a rolling window of the past 10 years is used to perform a new sorting into 25 portfolios. Stocks with the lowest betas are sorted into the first quintile. Then the stocks within this quintile are again sorted by co-skewness into quintiles. Stocks with betas in the second quintile are sorted by co-skewness into quintiles, and so on. Thus, there are 5 $\times$ 5 portfolios. The time series of stock returns within each portfolio are then used to find the betas and co-skewness measures. The sample estimates of beta and co-skewness of the formed equal-weighted portfolio returns versus the market return over the sampling period for each portfolio are equivalent to the equal-weighted averages of the individual stocks within each portfolio. As the average measures do not display significant variations across different sub-periods, we report only statistics for the first sub-period Jan 1936-Jun 1970, and for the overall period Jan 1936-Dec 2011.

	Sampling Period Jan 1936-Jun 1970		Sampling Period Jan 1936-Dec 2011	
Portfolio	Average beta	Average Co-skewness	Average beta	Average Co-skewness
1	0.5883	0.9961	0.5638	0.9707
2	0.7405	1.0596	0.6338	0.9091
3	0.7326	1.1451	0.6419	0.9993
4	0.7478	1.3454	0.6638	1.2652
5	0.8147	1.8330	0.7600	1.9434
6	1.0473	0.9346	0.9230	1.0966
7	1.0014	0.7286	0.9086	1.0547
8	1.0351	1.2312	0.9282	1.2786
9	1.0779	0.7940	0.9751	1.1658
10	1.1474	1.7711	1.0188	1.9795
11	1.3508	-0.6798	1.1445	0.4858
12	1.2541	0.5397	1.1337	0.9932
13	1.2365	1.2978	1.1239	1.4134
14	1.3227	1.0078	1.1586	1.4097
15	1.4189	1.3963	1.2083	1.8616
16	1.4862	-0.4313	1.2993	0.5591
17	1.4219	0.8706	1.2810	1.2279
18	1.4510	1.0573	1.3157	1.4013
19	1.4842	1.0664	1.3233	1.5218
20	1.5700	0.8933	1.3785	1.6217
21	1.7118	0.0729	1.5341	0.8581
22	1.7324	-0.3274	1.5464	0.7412
23	1.7641	0.5421	1.5958	1.1732
24	1.7182	0.8655	1.5710	1.4955
25	2.0030	1.2343	1.6730	1.9519

In Table 1, we show the average beta and co-skewness measures for each of the 25 portfolios formed in two sampling periods of January 1936-June 1970 (corresponding to (Kraus and Litzenberger, 1976)'s sampling period) and of January 1936 to December 2011 for the entire sample. The first 5 portfolios are those from the first quintile sorted by betas. Within these first 5 portfolios, they are then ranked by sorting in terms of gammas. Since these are ex-post measures they corresponded generally to increasing beta size and increasing gamma size though there would be minor variations from the ex-ante measures used to sort them.

Table 2: Table shows the average monthly excess market return rate, $\overline{R_{mt}-R_{ft}}$, the monthly volatility or standard deviation of market return rate, $\sqrt{\sum_{t=1}^{T}\left(R_{mt}-\overline{R_{mt}}\right)^2/(T-1)}$, and the monthly cube root of the third central moment of market return rate, $\sqrt[3]{\sum_{t=1}^{T}\left(R_{mt}-\overline{R_{mt}}\right)^3/(T-1)}$, for the various sub-periods. T refers to the sample size of the sub-period. R_{mt} and R_{ft} are the market return rate and the risk-free rate for period t. Sub-period 1: Jan 1936-Jun 1970, Sub-period 2: Jul 1970-Jun 2007, Sub-period 3: Jul 2007-Dec 2011, Sub-period 4: Jul 1970-Dec 2011, Total (overall) Period: Jan 1936-Dec 2011.

	$\overline{R_{mt}-R_{ft}}$	$\sqrt{\sum_{t=1}^{T}\left(R_{mt}-\overline{R_{mt}}\right)^2/(T-1)}$	$\sqrt[3]{\sum_{t=1}^{T}\left(R_{mt}-\overline{R_{mt}}\right)^3/(T-1)}$
Sub-Period 1	0.0100	0.0452	-0.0319
Sub-Period 2	0.0109	0.0426	-0.0273
Sub-Period 3	0.0066	0.0567	-0.0348
Sub-Period 4	0.0104	0.0444	-0.0296
Total Period	0.0103	0.0447	-0.0308

In Table 2, we show the average monthly excess market return rate, $\overline{R_{mt}-R_{ft}}$, the monthly volatility or standard deviation of market return rate, $\sqrt{\sum_{t=1}^{T}\left(R_{mt}-\overline{R_{mt}}\right)^2/(T-1)}$, and the monthly cube root of the third central moment of market return rate, $\sqrt[3]{\sum_{t=1}^{T}\left(R_{mt}-\overline{R_{mt}}\right)^3/(T-1)}$, for the various sub-periods of Jan 1936-Jun 1970, Jul 1970-Jun 2007, Jul 2007-Dec 2011, Jul 1970-Dec 2011, and total overall period of Jan 1936-Dec 2011. It is seen that during sub-period 3, July 2007 to December 2011, market return volatility has been the highest, and skewness is most negative. This is intuitively correct given that this period marked the global financial recession brought about by the collapse of the sub-prime mortgage market in U.S. and was followed by the European debt crisis since 2010.

In the models, theoretically $a_1 > 0$, $b_1 > 0$, $b_2 < 0$, $c_1 > 0$, $\alpha > 0$, and $a_0 = b_0 = c_0 = 0$. The results for the cross-sectional tests and the comparisons of the three models are shown in Table 3. We make the following list of observations of the results reported in Table 3. Recall that the discussed estimated coefficients are the averages across all ex-post monthly cross-sectional regressions.

(1) $\overline{\hat{a}_1}$ and $\overline{\hat{c}_1}$ are positive for all sub-periods in MCF and LM models. These are estimates of the market risk premium. Except for sub-period 3 during July 2007 to December 2011, most of the estimates are significantly higher than zero, especially in the LM model. Estimates of market risk premium by MCF and LM, however, are lower than the statistical averages of excess market returns shown in Table 2. Estimates of the market price of beta in KL model, $\overline{\hat{b}_1}$, however, are of the wrong signs, being significantly negative in sub-periods 2 and 4. When co-skewness or gamma tends to zero in a symmetric distribution, this price is the market risk premium, and should be positive.

(2) On the average $\overline{R^2}$, KL, MCF, and LM produce reasonably high explanation of the cross-sectional variation in decreasing order. The MCF model consistently performs better than LM model in this aspect.

Table 3: Table shows the average statistics of cross-sectional regressions. Sub-period 1: Jan 1936-Jun 1970, Sub-period 2: Jul 1970-Jun 2007, Sub-period 3: Jul 2007-Dec 2011, Sub-period 4: Jul 1970-Dec 2011, Total (overall) Period: Jan 1936-Dec 2011. * and $^+$ indicate significance at the 2-tail 1% and 5% level respectively.

MCF Model	$\overline{\hat{\theta}_0}$	$t(\overline{\hat{\theta}_0})$	$\overline{\hat{\theta}_1}$	$t(\overline{\hat{\theta}_1})$	$\overline{\hat{\theta}_2}$	$t(\overline{\hat{\theta}_2})$	$\overline{\hat{\theta}_3}$	$t(\overline{\hat{\theta}_3})$	Ave $\overline{R^2}$
Sub-Period 1	-0.0030	-1.1761	0.0038	2.9459*	-0.0037	-3.1531*	2.5087	180.32*	0.3225
Sub-Period 2	0.0027	1.6139	0.0147	1.6372	-0.0138	-1.9784+	3.2216	160.59*	0.3340
Sub-Period 3	-0.0009	-0.1021	0.0072	0.0727	-0.0048	-0.2426	14.5375	37.47*	0.4240
Sub-Period 4	0.0046	3.2441*	0.0291	2.0438+	-0.0216	-2.3604+	4.0099	158.07*	0.3373
Total Period	0.0010	0.7826	0.0048	3.1963*	-0.0055	-2.7079*	2.2603	156.15*	0.3119
KL Model	$\overline{\hat{\theta}_0}$	$t(\overline{\hat{\theta}_0})$	$\overline{\hat{\theta}_1}$	$t(\overline{\hat{\theta}_1})$	$\overline{\hat{\theta}_2}$	$t(\overline{\hat{\theta}_2})$			Ave $\overline{R^2}$
Sub-Period 1	-0.0076	-2.4102+	0.0104	2.9387*	-0.1785	-4.4297*			0.3633
Sub-Period 2	0.0003	0.1754	-0.0179	-3.7491*	-0.5311	-5.3733*			0.4092
Sub-Period 3	-0.0060	-0.6639	0.0052	0.6413	-0.0691	-0.4879			0.4198
Sub-Period 4	-0.0065	-2.6530*	-0.0152	-3.6383*	-0.6230	-5.6072*			0.4167
Total Period	-0.0111	-4.5512*	0.0042	2.0255+	-0.4010	-7.2878*			0.3975
LM Model	$\overline{\hat{\theta}_0}$	$t(\overline{\hat{\theta}_0})$	$\overline{\hat{\theta}_1}$	$t(\overline{\hat{\theta}_1})$	$\overline{\hat{\theta}_2}$	$t(\overline{\hat{\theta}_2})$			Ave $\overline{R^2}$
Sub-Period 1	-0.0057	-2.0238+	0.0365	4.6516*	-0.0579	-4.3704*			0.3728
Sub-Period 2	-0.0049	-1.9797	0.0746	5.4043*	-0.1285	-5.3552*			0.4074
Sub-Period 3	-0.0023	-0.2237	0.0019	0.0613	0.0017	0.0300			0.4448
Sub-Period 4	-0.0122	-3.6639*	0.0837	5.6698*	-0.1434	-5.6169*			0.4086
Total Period	-0.0112	-4.5724*	0.0685	7.4607*	-0.1144	-7.3327*			0.4084

(3) The KL model performed best for sub-period 1. In several sub-periods, it has intercepts significantly different from zero. MCF model also has two sub-periods that have intercepts different from zero. The intercepts should theoretically be zeroes. While KL model appears to overfit with higher average $\overline{R^2}$, MCF and LM models appear to be able to explain only partially the market premium, as the estimates based on the covariability and partial moment respectively, are lower.

There are possibly other systematic factors beyond market returns, such as are reported in (Fama and French, 1992).

(4) MCF model's estimates of $\overline{\hat{\alpha}}$, the reward for positive covariability, ranged from 0.3990 to 0.5733, all highly significantly positive. Thus the special case of Sharpe CAPM when $\alpha = 0$ may be said to be rejected. This positive $\overline{\hat{\alpha}}$ is consistent with the case of negative co-skewness premium, $\overline{\hat{b}_2}$ evident in the KL model. Moreover, it is interesting to note that the highest estimate of $\hat{\alpha}$ occurred in sub-period 3 during July 2007 to December 2011, the period of global financial crisis where market returns showed the most negative skewness as seen in Table 2. During this sub-period, it is intuitively correct that the reward for positive covariability should be highest. The $\hat{\alpha}$ estimates fall within $(0, 1)$, indicating the existence of a maximum expected utility with a unique computable optimal portfolio solution.

CONCLUSIONS

In this paper we propose a multiple criteria framework to model equilibrium asset pricing that includes the Sharpe-Lintner CAPM as a special case. While there are other developed models that also subsume the latter as a special case, such as the three-moment CAPM or the lower partial moment model, the generalized model in this paper carries all the advantages and prized classical results of (1) easy computation of optimal portfolio weights and thus establishment of uniqueness and existence of a maximum expected utility, (2) attainment of aggregation across heterogeneous agents with different risk aversions into a representative agent, and (3) properties of linear risk sharing and Pareto efficiency that comes with the usual linear aggregation result, that are often not available in the other models.

The original contribution in this paper is to show a portfolio variance decomposition into positive variability and non-positive variability, and to show how a new concept of positive covariability can be constructed as a reward for a general notion of positive skewness. It leads easily to a new systematic risk measure incorporating this covariability, and the new measure includes the usual CAPM beta as a special case.

We also provide several other motivations for the development of the ideas of positive variability and of positive covariability in this paper. These variables drive a general notion of positive skewness that is the basis for understanding how risk premium can be reduced from the level in the standard mean-variance model without considering other non-market systematic factors. In particular, the use of expected utility function incorporating positive variability, σ_p^{+2}, can be motivated by the idea of the role of options in stock valuation. It extends mean-variance pricing to a multi-criteria optimization framework using variance decomposition onto half-spaces.

The idea of variance decomposition and covariability may be extended to decompositions of quantiles. Such decompositions can be applied to study valuation of other types of assets that display returns asymmetry. The objects of positive or negative variabilities can possibly be used to construct new measures of asymmetry and higher moments.

In this paper our MCF efficient portfolio is shown to be empirically superior in performance to the received mean-variance portfolio, and appears to be more robust and stable than the skewness model as in (Kraus and Litzenberger, 1976). While somewhat similar in performance with the lower partial moment models of (Hogan and Warren, 1974) and of (Bawa and Lindenberg, 1977), our

MCF model nevertheless provides for more intuitive insights on the reduction of market risk premium via a covariability adjustment to the standard beta. All the three models are able to explain a sizeable portion of the cross-sectional return risk premium in the market. Like the Sharpe-Lintner CAPM and the lower partial moment model, the MCF extended two-moment model in this paper admits two-fund separation in the market, including a risk-free asset.

The implication of our result may be quite useful and important, as the mean-variance framework and the standard two-moment CAPM that have become industry household appliances in portfolio decisions, expected return calibrations, risk-adjusted discounting, and financial valuations of all sorts, can now be improved. This work builds upon extant asset pricing theories but provides for a very important extension.

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