USING A CONSTANT RATE TO APPROXIMATE A LINEARLY CHANGING RATE FOR THE EOQ AND EPQ WITH PARTIAL BACKORDERING

David W. Pentico, Palumbo-Donahue School of Business, Duquesne University, Pittsburgh, PA 15282-0180, pentico@duq.edu, 412-396-6252
Carl Toews, Dept. of Mathematics & Computer Science, University of Puget Sound, Tacoma, WA 98416, toewsc@gmail.com, 253-879-3839
Matthew J. Drake, Palumbo-Donahue School of Business, Duquesne University, Pittsburgh, PA 15282-0180, drake987@duq.edu, 412-396-1959

ABSTRACT

In order to gain some insight into how well partial backordering models for the EOQ and EPQ with a constant backordering rate can approximate the performance of the comparable models with a backordering rate that increases linearly with the time until the backorder can be filled, we conducted an experiment that compared the costs of using the two models over a reasonable set of parameter combinations. Our basic conclusion is that the simpler model with a constant backordering rate can perform virtually as well as the more complicated model as long as the critical value for the backordering rate is positive.

Keywords: EOQ with partial backordering, EPQ with partial backordering

INTRODUCTION

The early models for the basic deterministic economic order quantity with partial backordering (EOQ-PBO) developed by Montgomery et al. (1973), Rosenberg (1979), Park (1982,1983), and Wee (1989), one of the models included in San José et al. (2005), and the recent model by Pentico and Drake (2009) made all the usual assumptions of the basic deterministic EOQ model with full backordering except that they assumed that a constant percentage $\beta$ of the demand when there is no stock will be backordered, with the remaining percentage $1 - \beta$ being lost sales. The first models for the basic deterministic economic production quantity model with partial backordering (EPQ-PBO), by Mak (1987), Sharma and Sadiwala (1997) and Zeng (2001), and the recent model by Pentico et al. (2009), made the same assumption of a constant backordering rate, although Pentico et al. assumed that rate applied during the entire stockout interval, while Mak, Sharma and Sadiwala, and Zeng assumed that there was full backordering once production started again. Descriptions of all of these models, and many others, may be found in a survey by Pentico and Drake (2011).

The simplest models for the EOQ-PBO and the EPQ-PBO that do not assume that the backordering rate is a constant allow $\beta$ to change once during an inventory cycle. San José et al. (2005) included a step function for $\beta$ for the EOQ-PBO in which there is no backordering at all ($\beta = 0$) if the time until replenishment exceeds $a$ and then steps up to $\beta = 1$. Pentico et al. (2011) extended their EPQ-PBO model with a constant backordering rate to allow $\beta$ to increase when production starts, which includes the models by Pentico et al (2009), Mak (1987), Sharma and Sadiwala (1997) and Zeng (2001) as special cases, with Pentico et al. (2009) having the new backordering rate the same as the original one and the other three papers having $\beta$ increase to 1.
Wee and Tang (2012) extended the model in Pentico et al. (2011) to allow for a single change in \( \beta \) at a time other than when production begins.

The next simplest model type for the EOQ-PBO or the EPQ-PBO that does not assume that \( \beta \) is a constant is one in which the backordering rate is given by a linear function \( \beta(\tau) \), where \( \tau \) is the time remaining until the backorder can be filled, which is briefly discussed in Montgomery et al. (1973). More complete developments of the linear model for \( \beta(\tau) \) are given in San José et al. (2007) for the EOQ-PBO and Toews et al. (2011), which also considered the EPQ-PBO.

Other than the linear model for \( \beta(\tau) \) in Montgomery et al. (1973), the first models to include a backordering rate that increases over time were developed by Abad (1996), who assumed that \( \beta(\tau) \) is either an exponential function or a rational function of \( \tau \). Given their forms, in both of these functions \( \beta(\tau) \) increases as \( \tau \) decreases, approaching its maximum value, which is usually assumed to be 1.0, when \( \tau = 0 \). San José et al. (2006) developed a solution procedure when \( \beta(\tau) \) is an exponential function that is based on the same basic concepts as the approach they developed for other forms for \( \beta(\tau) \) in San José et al. (2005). Descriptions of the exponential and rational forms of \( \beta(\tau) \), along with the linear form and others proposed by other authors, are in Pentico and Drake (2011).

A significant problem with basic EOQ-PBO and EPQ-PBO models in which \( \beta(\tau) \) has any form other than a constant or a linear function of \( \tau \) is that they do not have a closed form solution. Solving the models for any other form for \( \beta(\tau) \) involves some sort of search process, usually either non-linear programming or some type of iterative process that involves a search procedure.

There are at least two problems with using these non-closed-form solution methods. First, they are more time-consuming and harder to automate. Second, and the primary impetus for the research reported here, is that they are more difficult for many, if not most, managers to understand. Why the difficulty of understanding how a model and/or its solution procedure works is a relevant issue for managing inventory can be summarized very succinctly by the following quote from Woolsey and Swanson (1975): “People would rather live with a problem they cannot solve than accept a solution they cannot understand.”

Here we consider the accuracy of approximating the EOQ-PBO and the EPQ-PBO with a backordering rate \( \beta(\tau) \) that is a linear function of \( \tau \), the time remaining until the backorder can be filled, by the EOQ-PBO or EPQ-PBO with a constant \( \beta = (1 + \beta_0)/2 \), where \( \beta_0 \) is the value of the linearly changing \( \beta(\tau) \) at the time the stockout begins. This is our first step in developing approximations for the more complicated situations in which \( \beta(\tau) \) is either an exponential or rational function of \( \tau \), scenarios that are not easily solved.

**NOTATION**

The notation for the parameters and variables to be used, which is basically the same as the notation used in Toews et al. (2011), is given in Table 1.
Table 1
Symbols Used and Their Meanings

Parameters

\(D\) = demand per year
\(P\) = production rate per year if constantly producing
\(s\) = the unit selling price
\(C_o\) = the fixed cost of placing and receiving an order
\(C_p\) = the variable cost of a purchasing or producing a unit
\(C_h\) = the cost to hold a unit in inventory for a year
\(C_b\) = the cost to keep a unit backordered for a year
\(C_g\) = the goodwill loss on a unit of unfilled demand
\(C_l = (s - C_p) + C_g\) = the cost for a lost sale, including the lost profit on that unit and any goodwill loss
\(\beta\) = the fraction of stockouts that will be backordered in a constant backordering rate model
\(\beta_0\) = the initial fraction of stockouts that will be backordered in a linearly changing backorder rate model
\(\beta(\tau)\) = the fraction of stockouts that will be backordered in a linearly changing backorder rate model
\(\tau\) = the time until the backorder will be filled

Variables

\(T\) = the length of an order cycle
\(F\) = the fill rate or the percentage of demand that will be filled from stock

SUMMARIES OF THE PARTIAL BACKORDERING MODELS

We begin with brief summaries of the models we will be using for the EOQ and EPQ with partial backordering, which are those used in Pentico and Drake (2009) for the EOQ-PBO with a constant \(\beta\), Pentico et al. (2009) for the EPQ-PBO with a constant \(\beta\), and Toews et al. (2011) for the EOQ-PBO and the EPQ-PBO with a linear function for \(\beta(\tau)\). In all of these models, as defined in Table 1, \(T\) is the length of an inventory cycle or the time between orders and \(F\) is the fill rate or percentage of demand filled immediately from stock.

**The EOQ-PBO with a constant backordering rate \(\beta\)**

This model makes all the assumptions of the basic EOQ with full backordering model except it assumes that only a given fraction or percentage \(\beta\) of the demand during the time that the system is out of stock is backordered, with the complementary fraction or percentage \(1 - \beta\) being lost sales. The average cost per period is:

\[
\Gamma(T,F) = \frac{C_o}{T} + \frac{C_b D T F^2}{2} + \frac{\beta C_b D T (1 - F)^2}{2} + C_l D (1 - \beta)(1 - F) \tag{1}
\]

As shown in Pentico and Drake (2009), the equations for the values of \(T\) and \(F\) that minimize the average cost per period are:
Pentico et al.  EOQ and EPQ-Partial Backordering-Approximations

\[ T^* = \sqrt{\frac{2C_o}{DC_h} \left[ \frac{C_h + \beta C_b}{\beta C_b} \right] - \frac{[(1-\beta)C_i]^2}{\beta C_h C_b}} \]  
(2)

\[ F^* = F(T^*) = \frac{(1-\beta)C_i + \beta C_b T^*}{T^*(C_h + \beta C_b)} \]  
(3)

only if \( \beta \) satisfies the condition given by Eq. (4):

\[ \beta \geq \beta^* = 1 - \sqrt{\frac{2C_o C_h D}{(DC_i)}} \]  
(4)

**The EPQ-PBO with a constant backordering rate \( \beta \)**

This model makes the same assumptions as the model for the basic EOQ-PBO with a constant \( \beta \) except that it makes the usual assumption of the basic EPQ that the delivery of the order is at a constant rate \( P \) rather than being instantaneous. The average cost per period is:

\[ \Gamma(T,F) = \frac{C_o}{T} + \frac{C_h DTF^2}{2} + \frac{\beta C_b DT(1-F)^2}{2} + C_i D(1-\beta)(1-F) \]  
(5)

where \( C_h = C(1-D/P) \) and \( C_b = C_b(1-\beta D/P) \). As a result, as shown in Pentico et al. (2009), the equations for the values of \( T \) and \( F \) that minimize the average cost per period are:

\[ T^* = \sqrt{\frac{2C_o}{DC_h} \left[ \frac{C_h + \beta C_b}{\beta C_b} \right] - \frac{[(1-\beta)C_i]^2}{\beta C_h C_b}} \]  
(6)

\[ F^* = F(T^*) = \frac{(1-\beta)C_i + \beta C_b T^*}{T^*(C_h + \beta C_b)} \]  
(7)

only if \( \beta \) satisfies the condition given by Eq. (8):

\[ \beta \geq \beta^* = 1 - \sqrt{\frac{2C_o C_h D}{(DC_i)}} \]  
(8)

which is identical to the condition given in Equation (4) if \( C_h \) replaces \( C_h \).

**The EOQ-PBO with a linear function for \( \beta(\tau) \)**

This model makes the same assumptions as the model for the basic EOQ-PBO with a constant \( \beta \) except that it assumes that \( \beta(\tau) \), the backordering rate \( \tau \) periods before the time at which the backorder will be filled, is a positive linear function of \( \tau \). As shown in Toews et al. (2011), the average cost per period is:

\[ \Gamma(T,F) = \frac{C_o}{T} + \frac{C_h DTF^2}{2} + \frac{C_b DT(1-F)^2}{2} + C_i D(1-F) \frac{(1-\beta_0)}{2} \]  
(9)

where \( C_b = C_b \left[ \beta_0 + \frac{(1-\beta_0)}{3} \right] \) and \( \beta_0 \) is the initial value of \( \beta(\tau) \) when the stockout begins. The equations for the values of \( T \) and \( F \) that minimize the average cost per period are:
\[ T^* = \sqrt{\frac{2C_o}{DC_h} \left[ \frac{C_h + C_b'}{C_b'} \right] - \left[ \frac{C_i(1 - \beta_0')/2}{C_h C_b'} \right]} \]  
\[ F^* = F(T^*) = \frac{C_i[(1 - \beta_0')/2] + C_b' T^*}{T^* (C_h + C_b')} \]

only if  \( \beta_0' \geq \beta_0^* = 1 - 2\sqrt{2C_0 C_h D / (DC_i)} \).  

Notice that the subtracted term in the equation for  \( \beta_0^* \) in Equation (12) is double the subtracted term in the equation for  \( \beta^* \) in Equation (4) for the model for the EOQ-PBO with a constant  \( \beta \), which makes sense since the average value of  \( \beta(\tau) \) is twice the value of  \( \beta_0 \).

The EPQ-PBO with a linear function for  \( \beta(\tau) \)

This model makes the same assumptions as the model for the basic EOQ-PBO with a linear function for  \( \beta(\tau) \) except that it makes the usual assumption of the basic EPQ that the delivery of the order is at a constant rate  \( P \) rather than being instantaneous. As shown in Toews et al. (2011), the average cost per period is:

\[ \Gamma(T,F) = \frac{C_o}{T} + \frac{C_i' D TF^2}{2} + \frac{C_b' DT (1 - F)^2}{2} + C_i D (1 - F) \left( \frac{1 - \beta_0}{2} \right) \]

where  \( C_h' = C(1 - D/P) \),  \( C_h' = C_b' \left[ \beta_0 + \frac{(1 - \beta_0)}{3} - \frac{(1 + \beta_0)^2 D}{4P} \right] \), and  \( \beta_0 \) is the initial value of  \( \beta(\tau) \) when the stockout begins. The equations for the values of  \( T \) and  \( F \) that minimize the average cost per period are:

\[ T^* = \sqrt{\frac{2C_o}{DC_h'} \left[ \frac{C_h' + C_b'}{C_b'} \right] - \left[ \frac{C_i(1 - \beta_0')/2}{C_h' C_b'} \right]} \]  
\[ F^* = F^*(T^*) = \frac{C_i[(1 - \beta_0')/2] + C_b' T^*}{T^* (C_h' + C_b')} \]

only if  \( \beta_0' \geq \beta_0^* = 1 - 2\sqrt{2C_0 C_h' D / (DC_i)} \)

which is identical to the condition given in Equation (12) if  \( C_h' \) replaces  \( C_h \).

THE STUDY

Our purpose is to evaluate the accuracy of approximating the EOQ-PBO and the EPQ-PBO with a backordering rate  \( \beta(\tau) \) that is a linear function of  \( \tau \), the time remaining until the order can be filled, by a constant  \( \beta \). We do this by examining the average and worst-case performance of the approximations on a set of test problems based on reasonable parameter values, which will help
identify the conditions under which the approximations perform less well and the conditions under which they can be expected to perform very well.

**Research Methodology**

Our performance measure is the ratio of the cost of using the solution from the constant-$\beta$ model to the cost of using the optimal solution for the linearly changing $\beta(\tau)$ model for a set of test problems based on reasonable values for five (six for the EPQ-PBO) situational characteristics. Four of the characteristics are basic problem parameters ($C_o$, $C_b$, $C_l/C_b$, and $D$). The fifth is the value of $\beta_0$, the initial value of $\beta(\tau)$, relative to $\beta_0^*$, the minimum value of $\beta_0$ for which partial backordering is optimal. The sixth characteristic added for the EPQ-PBO is $P/D$. All of these characteristics affect the values of $T^*$ and $F^*$. The values chosen for the parameters were selected to give a range of values for $\beta_0^*$ since this, as we shall see, has an effect on the average performance of the heuristics. For all the parameters except the $C_l/C_b$ ratio, the values used have “order of magnitude” differences. For $C_l/C_b$ the larger ratio is 2.5 times the lower ratio.

To determine the average cost ratios for the EOQ-PBO, the cost of the approximation will be obtained by substituting $T$ and $F$ from Eqs. (2) and (3) into Eq. (9). For the EPQ-PBO, the cost of the approximation will be obtained by substituting $T$ and $F$ from Eqs. (6) and (7) into Eq. (13).

**The Test Sets**

For both the EOQ and the EPQ, the values used for the first five situational characteristics were:

1. $C_o = 0.5, 5.0, 50$: This affects the value of $T^*$, with $C_o = 0.5$ leading toward JIT results. It also affects the value of $\beta_0^*$, with a smaller value for $C_o$ resulting in a larger $\beta_0^*$, which should improve the performance of the approximations.
2. $C_b = 0.5, 5.0$: $F^*$ should be lower for $C_b = 0.5$ (since backordering is less expensive) and higher for $C_b = 5.0$ (since backordering is more expensive) for the same value of $T^*$.
3. $C_l/C_b = 2.0, 5.0$: $C_l$ is important in determining the value of $\beta_0^*$, with a larger $C_l$ resulting in a larger $\beta_0^*$, which should result in better performance of the approximations.
4. $D = 20, 200$: A larger $D$ leads to a larger $\beta_0^*$ and, therefore, a narrower range for $\beta_0$ within which PBO is optimal. Thus a larger $D$ should improve the performance of the approximations for both the EOQ and the EPQ.
5. $\beta_0$ relative to $\beta_0^*$: Since the different combinations of the first four factors will usually give different values for $\beta_0^*$, $\beta_0$ for an experimental combination will be based on how large it is relative to $\beta_0^*$, rather than using fixed values for $\beta_0$. Three values of $\beta_0$ will be used: $\beta_0$ is 25% of the way between $\beta_0^*$ and 1.0, 50% of the way, and 75% of the way.
Because $\beta_0^* < 0$ may lead to negative values for $\beta_0$, we used $\max(\beta_0^*, 0)$ rather than $\beta_0^*$ when determining $\beta_0$’s value.

In all cases $C_h = 1$. This gives $3 \times 2 \times 2 \times 2 \times 3 = 72$ combinations of situational characteristics for testing the approximation for the EOQ.

For the EPQ comparisons, the values of the sixth characteristic were:

6. $P/D = 2, 20$: This was included because of its effect on $C_h$ and $C_b$, both of which are important in determining $T^*$ and $F^*$, while $C_h$ is important in determining $\beta_0^*$. Adding this sixth factor increases the number of test problems for the EPQ approximation to 144.

### Results for the Approximation for the EOQ-PBO when $\beta(\tau)$ Changes Linearly

Table 2 summarizes the test results for the approximation of the EOQ with a linearly changing $\beta(\tau)$ by the EOQ with a constant $\beta = (1 + \beta_0)/2$. Each row shows cost ratio information broken down by the value of $\beta_0$ relative to $\beta_0^*$ and 1.0. For example, EOQ(.25) means that row refers to the EOQ approximation with $\beta_0$ being 25% of the way from $\beta_0^*$ to 1.0. Looking at the first set of columns, we see that EOQ(.25) has an average cost ratio of 1.0006, with a maximum of 1.0056 and a minimum of 1.0000 for 24 cases. Examining the individual case results, we found that the highest cost ratios all came from the four cases for which $\beta_0^*$ is negative. These cases also resulted in the worst cost ratios for EOQ(.50) and EOQ(.75). The next set of columns

### Summary of Test Results for Approximations for the EOQ

<table>
<thead>
<tr>
<th>Group#</th>
<th>All Cases, n = 24</th>
<th>$\beta_0^* \geq 0$, n = 20</th>
<th>$\beta_0^* \geq 0.50$, n = 17</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Avg</td>
<td>Max</td>
<td>Min</td>
</tr>
<tr>
<td>EOQ(.25)</td>
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<td>1.0056</td>
<td>1.0000</td>
</tr>
<tr>
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<tr>
<td>All</td>
<td>1.0003</td>
<td>1.0056</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

# This refers to the position of $\beta_0$ relative to $\beta_0^*$ and 1.0.

contains the summaries of the results for the 20 combinations of $C_o, C_b, C_h$ and $D$ for which $\beta_0^* \geq 0$. For those cases the average ratio, ignoring the relative position of $\beta_0$, was less than 1.0001, with a worst case less than 1.0004. Referring to the final three columns, if we limit our attention to the 17 parameter combinations and 51 total cases for which $\beta_0^* \geq 0.50$, we find that the average ratio was less than 1.00005, with a maximum of 1.0001. Thus, as long as $\beta_0^*$ is positive,
we can expect the approximation to perform well, and if \( \beta^*_0 \) is at least 0.5, it can be expected to perform essentially as well as the EOQ-PBO with linearly changing \( \beta(\tau) \).

Looking at the individual case results summarized in Table 2 in more detail, we see two things of interest that are not at all surprising:

- For any given value for \( \beta^*_0 \), the lowest ratios occur when the value of \( \beta_0 \) is closer to 1.0.
- Looking at the cost ratio information broken down by the value of \( \beta_0 \) relative to \( \beta^*_0 \), the cost ratio is closer to 1.0 the closer \( \beta^*_0 \) is to 1.0.

Neither of these results is a surprise because they stem from the same basic idea: The closer \( \beta_0 \) is to 1.0, the less change there will be in the value of \( \beta(\tau) \) as \( \tau \) decreases. Thus, the less difference there will be between a linearly changing \( \beta(\tau) \) and a constant \( \beta \).

Since the average and highest cost ratios for all 72 total cases are only 1.0006 and 1.0056 respectively, it is not surprising that there is very little difference in the approximation’s performance when different parameter values are considered.

1. Grouped by \( C_o \): Since a lower value of \( C_o \) leads to a shorter inventory cycle, the average (maximum) ratios are, as expected, lowest for \( C_o = 0.5 \) at 1.0000 (1.0001), in the middle for \( C_o = 5.0 \) at 1.0004 (1.0026) and highest for \( C_o = 50 \) at 1.0015 (1.0056).

2. Grouped by \( C_b \): Since a lower (higher) value of \( C_b \) makes backordering more (less) attractive, it should be expected that the approximation’s performance would be better for \( C_b = 2.5 \) than for \( C_b = 0.5 \), which it is.

3. Grouped by the \( C_l/C_b \) ratio: Since a higher cost for a lost sale makes backordering less attractive, it should be expected that the approximation’s performance would be better for \( C_l/C_b = 2 \) than it is for \( C_l/C_b =5 \), which it is. The average (maximum) ratios are 1.0002 (1.0033) for \( C_l/C_b = 2 \) and 1.0005 (1.0056) for \( C_l/C_b = 5 \).

4. Grouped by \( D \): As noted above, a larger \( D \) leads to a larger \( \beta^*_0 \) and, therefore, a narrower range for \( \beta_0 \) within which PBO is optimal. Thus a larger \( D \) should improve the performance of the approximation, which it does. The average (maximum) ratios are 1.0005 (1.0056) for \( D = 20 \) and 1.0001 (1.0026) for \( D = 200 \).

5. Grouped by the ratio of \( \beta_0 \) to \( \beta^*_0 \): For all combinations of the first four parameters, the lowest cost ratio was found when \( \beta_0 \) was 75 percent of the way from \( \beta^*_0 \) to 1.0. For all combinations in which \( \beta^*_0 \) is less than 0.6, the highest ratio was found when \( \beta_0 \) is 25 percent of the way from \( \beta^*_0 \) to 1.0. If \( \beta^*_0 \) is more than 0.6, the highest ratio is found when \( \beta_0 \) is 50 percent of the way from \( \beta^*_0 \) to 1.0; in all of these combinations the cost ratio for all three values for \( \beta_0 \) is less than 1.00004, which means that there is no real difference between the cost ratios for the three starting values when \( \beta^*_0 > 0.6 \).

**Results for the approximation for the EPQ-PBO when \( \beta(\tau) \) changes linearly**

Table 3 summarizes the results of the tests for the EPQ-PBO. As in Table 2, each row shows cost ratio information broken down by the value of \( \beta \) relative to \( \beta^* \) and 1.0. Although the cost
ratios for the approximation are not as good for the EPQ as they are for the EOQ, most of the same conclusions can be drawn.

Table 3

Summary of Test Results for Approximations for the EPQ

<table>
<thead>
<tr>
<th>Group#</th>
<th>All Cases, n = 48</th>
<th>$\beta_0^* &gt; 0$, n = 42</th>
<th>$\beta_0^* &gt; 0.50$, n = 36</th>
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<tbody>
<tr>
<td></td>
<td>Avg</td>
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<td>EOQ(.25)</td>
<td>1.0026</td>
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</tr>
<tr>
<td>All</td>
<td>1.0015</td>
<td>1.0253</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

# This refers to the position of $\beta_0$ relative to $\beta_0^*$ and 1.0.

Breaking down the results on the basis of the values of $C_o$, $C_b$, $C_l/C_b$, $D$ and the size of $\beta_0$ relative to $\beta_0^*$ for the EPQ shows basically the same thing as it did for the EOQ, although the differences for the EPQ are a little bigger since the average and maximum ratios for the 144 cases are 1.0015 and 1.0253 respectively.

1. Grouped by $C_o$: The average (maximum) ratios are, as expected, lowest for $C_o = 0.5$ at 1.0001 (1.0009), in the middle for $C_o = 5.0$ at 1.0009 (1.0124) and highest for $C_o = 50$ at 1.0036 (1.0253).

2. Grouped by $C_b$: Since a lower (higher) value of $C_b$ makes backordering more (less) attractive, it should be expected that the approximation’s performance would be better for $C_b = 2.5$ than for $C_b = 0.5$, which it is. For $C_b = 2.5$ the average (maximum) is 1.0002 (1.0031) versus 1.0029 (1.0253) for $C_b = 0.5$.

3. Grouped by the $C_l/C_b$ ratio: Since a higher cost for a lost sale makes backordering less attractive, is should be expected that the approximation’s performance would be better for $C_l/C_b = 2$ than it is for $C_l/C_b = 5$, which it is. The average (maximum) ratios are 1.0008 (1.0157) for $C_l/C_b = 2$ and 1.0023 (1.0253) for $C_l/C_b = 5$.

4. Grouped by $D$: As noted above, a larger $D$ leads to a larger $\beta_0^*$ and, therefore, a narrower range for $\beta_0$ within which PBO is optimal. Thus a larger $D$ should improve the performance of the approximation, which it does. The average (maximum) ratios are 1.0025 (1.0253) for $D = 20$ and 1.0006 (1.0124) for $D = 200$.

5. Grouped by the value of $\beta_0$ relative to $\beta_0^*$: For all 48 combinations of the first four parameters and $P/D$, the lowest cost ratio occurs when $\beta_0$ is 75 percent of the way from $\beta_0^*$ to 1.0. The issue of which value of $\beta_0$ gives the highest cost ratio is not quite as clear for the EPQ as it is for the EOQ.

- For $P/D = 20$, the result is the same as it is for the EOQ: for all combinations in which $\beta_0^*$ is less than 0.6, the highest ratio occurs when $\beta_0$ is 25 percent of the way from $\beta_0^*$ to 1.0; if $\beta_0^*$ is more than 0.6, the highest ratio occurs when $\beta_0$ is 50 percent...
of the way from $\beta^*_0$ to 1.0, but in all of these combinations with a higher value for $\beta^*_0$
the cost ratio for all three values for $\beta_0$ is less than 1.00032, which means that there is
no real difference between the cost ratios for the three starting values when $\beta^*_0 > 0.6$.

- For $P/D = 2$, the result is almost the same: for all combinations except one in which $\beta^*_0$ is less than or equal to 0.6, for which $\beta^*_0 = 0.3675$, the highest ratio occurs when $\beta_0$ is 25 percent of the way from $\beta^*_0$ to 1.0; if $\beta^*_0$ is more than 0.6, the highest ratio is
found when $\beta_0$ is 50 percent of the way from $\beta^*_0$ to 1.0, but in all of these combinations with a higher value for $\beta^*_0$, the cost ratio for all three values for $\beta_0$ is
less than 1.00041, which means that there is no real difference between the cost ratios
for the three starting values when $\beta^*_0 > 0.6$.

6. Grouped by $P/D$: In addition to the results just observed in which the $P/D$ ratio makes a
small difference in determining which value of $\beta_0$ gives the largest cost ratio, we observe
the following: For all 72 combinations of the values of the first five experimental factors,
the cost ratio with $P/D = 2$ is lower than it is for the same case with $P/D = 20$. This is not
surprising since the effect of $P/D = 2$ is to reduce the effective holding cost per unit by
half, while with $P/D = 20$ it is only reduced by five percent. Among other things, the
lower value of $C_h^*$ leads to a higher value for $\beta^*_0$, which typically results in better
performance for both approximations.

SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

In order to gain some insight into using a constant $\beta$ to approximate a backordering-rate function $\beta(\tau)$ that is based on $\tau$, the time until the backorder can be filled, we have evaluated the accuracy of approximating the EOQ-PBO and the EPQ-PBO with a backordering rate $\beta(\tau)$ that is a linear function of $\tau$ by the comparable model with a constant $\beta$. We do this by examining the average and worst-case performance of the approximations on a set of test problems based on reasonable parameter values, which will help identify the conditions under which the approximations perform less well and the conditions under which they can be expected to perform very well.

Our basic conclusion is that, while differences in the situational parameter values have some
impact on the performance of the approximations, in most cases the approximations are
extremely good, costing a small fraction of a percent more than using the model with a time-
based $\beta(\tau)$. At worst, the cost for the EOQ approximation was less than 0.6 percent higher than
the cost from the model with a linearly changing $\beta(\tau)$ and for the EPQ approximation it was no
more than about 2.5 percent, and these worst-case results occurred only if $\beta^*_0$ was negative.

The high quality of the results in this experiment obtained by using the simpler model, which has
a closed-form solution, suggests that there is great potential for using either a constant-$\beta$ EOQ-
PBO or EPQ-PBO model or the comparable model with a linearly changing backorder-fraction $\beta(\tau)$, both of which also have closed-form solution equations, to do well in approximating the results from models for the EOQ-PBO and EPQ-PBO with $\beta(\tau)$ having an exponential or rational
form, both of which have $\beta(\tau)$ increasing in a non-linear fashion as $\tau$ decreases and both of which require more complicated solution procedures.

What will make conducting experiments to evaluate these approximations more difficult than what we have done here is the following: an important characteristic of the EOQ-PBO and EPQ-PBO with a linear function for $\beta(\tau)$ is the assumption that it is possible to determine $\beta_0$, the initial value for $\beta(\tau)$ when the stockout begins. This makes it possible to determine an average value for $\beta(\tau)$ over the stockout interval. The models using an exponential or rational form for $\beta(\tau)$ do not make this assumption. Instead, they just specify how the value of $\beta(\tau)$ decreases as $\tau$ increases, in both cases gradually approaching 0 as $\tau$ gets arbitrarily large. To use either the linearly changing form of $\beta(\tau)$ or a constant $\beta$, it will be necessary to develop a method for estimating $\beta_0$ or the average value of $\beta$ over the stockout interval.

REFERENCES


