VALUATION MODEL WITH CLOSED-FORM SOLUTIONS TO SIMULATE THE EFFECTS OF NON-DIVERSIFIABLE JUMPS IMPOSED ON THE VALUE OF DERIVATIVES

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ABSTRACT

The non-diversifiable jump risk is fatal to economic security. However, The traditional pricing theory of financial derivatives focused mostly on derivatives dependent on prices of traded assets, and seldom cared about other influence caused by other uncertain factors (such as climate, temperature or unknown effects of greenhouse gas (GHG) emission reductions, etc) that are not tradable on securities markets. But actually, the prices of derivatives will be changed under the influence of climate and other multiple correlated or uncorrelated factors (Some climatic or weather factors, such as air temperature, does affect the price of some derivatives, and certainly are underlying variables of derivatives per se). Some of these factors will cause systematic risks that can not be diversified away. Correspondingly, in this paper, we mainly introduce a unified method to discuss derivatives pricing problems under the circumstances mentioned previously, and deduce a general valuation model for the derivatives whose underlying variables are not traded assets, but with non-diversifiable risk. And this methodology has been extended to the situation when the underlying state variables change with jumps, i.e., the state variables may reveal sudden and rare breaks logically accounted for by exogenous events on information. We also study the pricing model when the equations of underlying state variables are nonlinear and derive a general valuation equation for derivatives. Furthermore, as compared with Merton [1976, 1994] and other scholars’ work, the unified pricing methodology given by us studies not only diversifiable jump risk, but also non-diversifiable jump risk, i.e., this valuation methodology can be used to cope with diversifiable jump risk which is nonsystematic as well as non-diversifiable jump risk which is systematic. And we also introduce a key term, which was ignored by former researchers of derivatives pricing, to demonstrate the tremendous effects of non-diversifiable jump imposed on the derivatives’ value. Finally, closed form solutions are deduced under some conditions.

Keywords: Uncertain factor, State variable, Derivatives, General valuation (or pricing) equation, Non-diversifiable jump risk

INTRODUCTION

The theory of valuation of financial derivatives has been developing very rapidly. But in most cases, the underlying variables of derivatives are often the prices of investment assets. Few other factor such as temperature, which is something far removed from financial markets, would be treated as the underlying variables. However, there are some financial derivatives whose underlying variables are those factors for sure. We can not ignore those factors since they may cause non-diversifiable risks, and we should have to study a valuation model for derivatives which will reflect the effects of these non-diversifiable risks.

John C. Hull has studied this problem systematically in his work Options, futures, & other
derivatives [Hull, 2000, 2003, 2012]. He discussed the differential equation satisfied by the price \( f \) of a derivative that is dependent on \( n \) underlying state variables \( \theta_l, \ l = 1, \cdots, n \). The process followed by \( \theta_l \) is

\[
d\theta_l = m_l \theta_l dt + s_l \theta_l dz_l.
\]

The state variables \( \theta_l, \ l = 1, \cdots, n \), are not necessarily the prices of investment assets, they can be something as far removed from financial markets as the temperature in other place. Thus we cannot construct an instantaneously riskless portfolio of Black-Scholes style, which consists of \( \theta_l \) and \( f \). Furthermore, we cannot duplicate \( f \) with portfolio of \( \theta_l \) either. John C. Hull introduced a new procedure to form an instantaneously riskless portfolio, using the derivative securities. But he only studied under the condition that \( \theta_l \) is continuous with respect to \( t \), and didn’t prove the uniqueness of \( \lambda_l \) in equation \( \mu - r = \sum_{l=1}^{n} \lambda_l \sigma_l \), where \( \lambda_l \) is interpreted as market price of risk for \( \theta_l \). Here we should emphasize that the proof of uniqueness of array \( \lambda_l \) is necessary. If the uniqueness of \( \lambda_l \) is not true, then some contradictions may arise and ambiguous meaning may be caused. In this paper, we prove the uniqueness of array \( \lambda_l \) in equation \( \mu - r = \sum_{l=1}^{n} \lambda_l \sigma_l \).

According to Merton’s point of view [1976, 1994], the total change in the stock price is posited to be the composition of two types of changes: (1) The normal vibrations in price, for example, due to a temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook, or other new information that causes marginal changes in the stock’s value. In essence, the impact of such information per unit time on the stock price is to produce a marginal change in the price (almost certainly). This component is modeled by a standard geometric Brownian motion with a constant variance per unit time and it has a continuous sample path. In this situation, there is ample scope for the famous Black-Scholes formula. (2) The abnormal vibrations in price are due to the arrival of important new information about the stock that has more than a marginal effect on price. Usually, such information will only be specific to the firm or possibly its industry and may have little impact on the market in general. Thus this kind of information will just cause nonsystematic risk, which will be diversifiable. But some information might bring disasters in the world as well as in the financial markets. And consequently, the corresponding risk may not be thought as diversifiable one.

It is reasonable to expect that there will be active times in the stock when such information arrives and quiet times when it dose not arrive although the active and quiet times are random. According to its very nature, important information arrives only at discrete points of time. This component is modeled by a jump process reflecting the non-marginal impact of the information. (This jump process can also be considered as a model reflecting the sudden breaks). And on this occasion of abnormal vibrations, the Black-Scholes formula is not valid, even in the continuous limit, because the stock price dynamics cannot be represented by a stochastic process with a continuous sample path. We must use stochastic differential equations with jumps to study those problems. And it is reasonable that the abnormal vibrations in state variables (not necessarily the prices of investment assets) are due to the arrival of important new information about them that has more than a marginal effect on those state variables. We introduce \( n \) Wiener process \( z_l, \ l = 1, \cdots, n \), and one Poisson jump process \( Q \) to describe \( \theta_i, \ i = 1, \cdots, n + 1 \). On basis of these, we deduce the valuation
equation of \( f \), and modify something not rigorous on Hull’s book.

Although Merton made an assumption that the jump component of the asset’s return represents nonsystematic risk (i.e., diversifiable risk not priced in the economy), the sudden breaks to the financial markets in the real world may be systematic and non-diversifiable risk. For example, the sort of risk of the Sept. 11 attacks, and the Financial Tsunami (caused by the sub-prime mortgage crisis), which resulted in disasters in the world as well as in the financial markets, certainly belong to non-diversifiable risk. The investor can not avoid meeting this kind of risk whose market price of risk is not zero. Thus they should be compensated more risk premium over normal return for suffering from this abnormal non-diversifiable risk. The valuation methodology to cope with this risk is also necessary in this situation. Wang, S.-Y. and Lin, S.-K. have given some analysis in some pricing problems with systematic jump risk [2010].

In the following discussion, we introduce a unified new method to price a derivative whose underlying state variables are not necessarily the prices of investment assets and bear abnormal non-diversifiable risk of jumps, especially revealing the effects of these non-diversifiable risks by a key term which has been neglected by other researchers [Barhan, Ikeda, Merton, Mizrach, Picquè, Wang]. Furthermore, we study the about problems in nonlinear case, i.e., when the equations of underlying state variables are nonlinear. Finally, we get closed form solutions under some conditions.

**STATE VARIABLES AND DERIVATIVES**

Suppose that there are a total of at least \( n + 1 \) traded derivative securities (including the one under consideration. Some of them may be fictitious) whose prices

\[ f_j = f_j(t, \theta_1, \ldots, \theta_n) = f_j(t, \theta), \quad j = 1, \ldots, n + 1, \]

depend on some or all of the \( n \) state variables \( \theta_i, \quad i = 1, \ldots, n \) and time \( t \), where we denote that \( \theta \triangleq (\theta_1, \ldots, \theta_n) \). \( f_j \) is twice continuous differentiable with respect to \( \theta_i, \quad i = 1, \ldots, n \) and once continuous differentiable in \( t \). And assume that the Jacobi matrix

\[ \frac{\partial f_j}{\partial \theta_i} \]

of \( f_j \) with respect to \( \theta_i \) has full rank, and other conditions introduced by John C. Hull [2000, 2003, 2012] still hold.

The state variables \( \theta_i \) are dominated by the following equations

\[ d\theta_i = m_i \theta_i dt + \theta_i \sum_{l=1}^{n} s_{il} dz_l = m_i \theta_i dt + \theta_i s_i dz, \quad (1.1) \]

i.e., each \( \theta_i \) is subject to the influence of \( z_l (1 \leq l \leq n) \), and \( \theta_i \neq 0 \). In equation (1.1), \( z_l, \quad 1 \leq l \leq n \) are standard Wiener processes, \( m_i \) is the expected growth rate in \( \theta_i \), and \( s_{il} \) is the volatility of the influence of \( z_l \) on \( \theta_i \), which forms a nondegenerate matrix \( s = (s_{il})_{n \times n} \), where \( s_i = (s_{i1}, \ldots, s_{in}) \) denotes the \( i \)th row vector of \( s \), and \( dz = (dz_1, \ldots, dz_n)^T \) is the \( n \) dimensional column vector of Wiener processes. \( m_i \) and \( s_{il} \) can be functions of any of the \( n \) state variables \( \theta_1, \ldots, \theta_n \) and time \( t \). Other notation is listed as follows:

- \( \rho_{il} \): Correlation between \( dz_i \) and \( dz_l \) \( (1 \leq i, l \leq n) \);
- \( \rho = (\rho_{il})_{n \times n} \): \( n \times n \) Correlation matrix between \( n \) dimensional stochastic vectors \( dz \) and \( dz \);
- \( r \): Instantaneous risk-free rate.
Applying Itô’s rule to \( f_j \), and taking (1.1) into account, we have

\[
df_j = \frac{\partial f_j}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f_j}{\partial \theta_i} d\theta_i + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \left\langle d\theta_i, d\theta_k \right\rangle
\]

\[
= \frac{\partial f_j}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \theta_i \theta_k \sigma_{il} \rho_{il}^T \right) dt + \sum_{l=1}^{n} \left( \sum_{i=1}^{n} \frac{\partial f_j}{\partial \theta_i} s_{il} \right) dz_l
\]

\[
\triangleq \mu_j f_j dt + \sum_{l=1}^{n} \sigma_{jl} f_j dz_l,
\]

where

\[\mu_j f_j = \frac{\partial f_j}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \theta_i \theta_k \sigma_{il} \rho_{il}^T ,\]

\[\sigma_{jl} f_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial \theta_i} \theta_i \sigma_{il} .\]  

In these equations, \( \mu_j \) is the instantaneous expected rate of return provided by \( f_j \) and \( \sigma_{ij} \) is the component of volatility of the return provided by \( f_j \). \( \mu_j \) and \( \sigma_{ij} \) suffer from the influences of \( \theta_1, \cdots, \theta_n \). We should remark that the main risks suffered by each \( \theta_i \) are the all Brownian motion \( z_l, \ l = 1, \cdots, n \), it means that the main uncertainties should be attributed to the Brownian motion \( z_l, \ l = 1, \cdots, n \). Virtually, \( \sigma_{ij} \) stands for the volatility caused by stochastic influence of the Brownian motion \( z_l \) on \( f_j \). In addition, the Brownian motion \( z_l \) affect derivative \( f_j \) via the direct influences of \( z_l \) on \( \theta_1, \cdots, \theta_n \).

Although we cannot make a portfolio with state variables \( \theta_1, \cdots, \theta_n \), since they are not necessarily the prices of investment assets, we can form an instantaneously riskless portfolio \( \Pi \) by means of \( n + 1 \) traded securities,

\[\Pi = \sum_{j=1}^{n+1} x_j f_j .\]

Here \( x_j \) is the amount of the \( j \)th security in the portfolio, we try to choose proper \( x_j \) that are not all zero, such that the stochastic components \( dz_l \) are eliminated. From equation (1.2), this means that

\[\sum_{j=1}^{n+1} x_j \sigma_{lj} f_j = 0,\]

\[l = 1, \cdots, n .\] Then the return from the portfolio becomes

\[d\Pi = \sum_{j=1}^{n+1} x_j \mu_j f_j dt .\]

Since there are no arbitrage opportunities, the portfolio must earn the risk-free return rate, so that
\[ d\Pi = r\Pi dt. \quad (1.8) \]

Take (1.5) and (1.7) into consideration, we arrive to
\[ \sum_{j=1}^{n+1} x_j \mu_j f_j = r \sum_{j=1}^{n+1} x_j f_j, \quad (1.9) \]
that is
\[ \sum_{j=1}^{n+1} x_j f_j (\mu_j - r) = 0. \quad (1.10) \]

(1.6) and (1.10) form a system of \( n + 1 \) simultaneous homogeneous linear equations with respect to \( x_j \). It follows from (1.4) that the coefficient matrix of the equations (1.6) can be written as
\[
\begin{pmatrix}
\sigma_{11} f_1 & \sigma_{12} f_2 & \cdots & \sigma_{1n} f_n & \sigma_{1,n+1} f_{n+1} \\
\sigma_{21} f_1 & \sigma_{22} f_2 & \cdots & \sigma_{2n} f_n & \sigma_{2,n+1} f_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{n1} f_1 & \sigma_{n2} f_2 & \cdots & \sigma_{nn} f_n & \sigma_{n,n+1} f_{n+1}
\end{pmatrix} = \begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_{n+1}
\end{pmatrix} = \begin{pmatrix}
\sigma y f_j \big|_{n \times (n+1)} = \sum_{i=1}^{n} \left( \frac{\partial f_j}{\partial \theta_i} \right) \theta_i s_{il}
\end{pmatrix}_{n \times (n+1)}
\]

\[
= \begin{pmatrix}
\frac{\partial f_1}{\theta_1} & \frac{\partial f_1}{\theta_2} & \cdots & \frac{\partial f_1}{\theta_n} \\
\frac{\partial f_2}{\theta_1} & \frac{\partial f_2}{\theta_2} & \cdots & \frac{\partial f_2}{\theta_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_{n+1}}{\theta_1} & \frac{\partial f_{n+1}}{\theta_2} & \cdots & \frac{\partial f_{n+1}}{\theta_n}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_n
\end{pmatrix}
\begin{pmatrix}
s_{11} & s_{12} & \cdots & s_{1n} \\
s_{21} & s_{22} & \cdots & s_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
s_{n1} & s_{n2} & \cdots & s_{nn}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{\partial f_j}{\theta_i} & 0 & \cdots & 0 \\
0 & \theta_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \theta_n
\end{pmatrix}
\begin{pmatrix}
s_{ij} \big|_{n \times n}
\end{pmatrix}
\]

\[ (1.11) \]

where \( l = 1, \cdots, n \), \( j = 1, \cdots, n + 1 \). Since Jacobi matrix \( \left[ \frac{\partial f_j}{\theta_i} \right]_{n \times n} \) has full rank, \( s = (s_{ij})_{n \times n} \) is nondegenerate, \( \theta_i \neq 0, \ i = 1, \cdots, n \), the coefficient matrix of the equations (1.6), which is the first matrix in (1.11), also has full rank. Then we know that the row vectors of the coefficient matrix of the equations (1.6) are linear independent. In order to make (1.6) and (1.10), the simultaneous homogeneous linear equations, have non-zero solutions, we must let the row vectors of coefficients of (1.6) and (1.10) be linear dependent. It follows from a well-known theorem in linear algebra [cf. Zhao, 1988] that the row vectors of the coefficient of (1.10) can be uniquely linear represented by the row vectors of the coefficient of (1.6). I.e., there exists a unique array of \( \lambda_l, \ l = 1, \cdots, n \), that are dependent only on the state variables \( \theta_i, \ i = 1, \cdots, n \) and time such that
\[ f_j (\mu_j - r) = \sum_{l=1}^{n} \lambda_l \sigma y f_j, \quad (1.12) \]
or
\[ \mu_j - r = \sum_{l=1}^{n} \lambda_l \sigma_{jl} . \]  
\hspace{1cm} (1.13)

Dropping the subscripts to \( f_j \), we reach
\[ df = \mu f dt + \sum_{l=1}^{n} \sigma_l f dz_l , \]  
\hspace{1cm} (1.14)

and
\[ \mu - r = \sum_{l=1}^{n} \lambda_l \sigma_l , \]  
\hspace{1cm} (1.15)

where
\[ \mu f = \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial f}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} \theta_i \theta_k s_i s_k^T , \]  
\hspace{1cm} (1.16)

\[ \sigma_l f = \sum_{i=1}^{n} \frac{\partial f}{\partial \theta_i} \theta_i s_{il} . \]  
\hspace{1cm} (1.17)

\( \lambda_l \) can be thought as the market price of risk produced by \( z_l \) in (1.1). (Each \( \theta_i , i = 1, \cdots, n \) are under the influence of all \( z_l \), \( l = 1, \cdots, n \)). And \( \lambda_l \sigma_l \) is the risk premium to compensate the investors for the risk arising from \( z_l \), where \( z_l \) affects derivative \( f \) via the direct influences of \( z_l \) on \( \theta_1, \cdots, \theta_n \).

So far, we have modified the conclusion of John C. Hull [2000, 2003, 2012], by proving the uniqueness of \( \lambda_l \), and giving a more general result. We should emphasize that the proof of uniqueness of array \( \lambda_l \) is necessary. If the uniqueness of array \( \lambda_l \) is not true, then there is some other array \( \lambda_l ^* \), which may not be market price of risk produced by \( z_l \), but yet satisfies equation (1.15) (i.e., without the uniqueness of array \( \lambda_l \), we cannot exclude the possibility that some array \( \lambda_l ^* \), not the market price of risk produced by \( z_l \), satisfies equation (1.15)). Thus some contradictions and ambiguous meaning may arise.

**DERIVATION OF THE GENERAL PRICING EQUATION OF DERIVATIVES**

From (1.15) we have
\[ \mu f - rf = \sum_{l=1}^{n} \lambda_l \sigma_l f . \]  
\hspace{1cm} (2.1)

Substituting (1.16) and (1.17) into (2.1), comes the equation
\[ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial f}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} \theta_i \theta_k s_i s_k^T - rf = \sum_{l=1}^{n} \lambda_l \sum_{i=1}^{n} \frac{\partial f}{\partial \theta_i} \theta_i s_{il} , \]  
\hspace{1cm} or
\[
\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \theta_i \frac{\partial f}{\partial \theta_i} \left( m_i - \sum_{i=1}^{n} \lambda_i s_i \right) + \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} \theta_i \theta_k s_i s_k^T = rf .
\] (2.2)

Thus we get some similar but more sophisticated conclusions, referring to John C. Hull [2000, 2003, 2012]. All these procedures can be extended to the situation of jumps, i.e., the underlying state variables are not continuous. We will discuss it in the next section.

**GENERAL PRICING EQUATION OF DERIVATIVES WHOSE UNDERLYING VARIABLES SUBJECT TO SUDDEN AND RARE BREAKS PRODUCING NON-DIVERSIFIABLE JUMP RISKS**

In this section, let’s introduce the Poisson jump process \( Q \) first [Barhan, Ikeda, Merton, Mizrahi, Picqué, Wang]. Supposed that the state variables \( \theta_i, \ i = 1, \cdots, n + 1 \) satisfies the equations

\[
d\theta_i(t) = \theta_i(t-) \left( s_{ii} dt + \sum_{i=1}^{n} s_{ii} dz_i + a_i dQ(t) \right) = \theta_i(t-) \left( m_i dt + s_i dz + a_i dQ(t) \right),
\] (3.1)

where \( dz = (dz_1, \cdots, dz_n)^T \), and \( s_{ii} \) is the volatility of the influence of \( z_i \) on \( \theta_i \), \( s = (s_{ii})_{(n+1)\times n} \) is a matrix with full rank. \( s_i = (s_{i1}, \cdots, s_{in}) \) denotes the \( i \)th row vector of \( s \), \( a_i \) is the volatility of jump, \( Q(t) = N(t) - \int_0^t \Lambda(s) ds \), \( N(t) \) is a Poisson counting process, \( \Lambda(t) \) is the intensity of jump. Now we introduce an assumption that there are at least \( n + 2 \) traded derivative securities \( f_j = f_j(t, \theta_1, \cdots, \theta_{n+1}) \), \( j = 1, \cdots, n + 2 \), where \( \theta \triangleq (\theta_1, \cdots, \theta_{n+1}) \). Applying to \( f_j \) Itô’s rule [Ikeda] with jumps, with the help of (3.1), and noting \( \bar{\theta} \triangleq (\theta_1(1 + a_1), \cdots, \theta_{n+1}(1 + a_{n+1})) \), we have

\[
df_j = \frac{\partial f_j}{\partial t} dt + \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} \theta_i \left( m_i dt + \sum_{i=1}^{n} s_{ii} dz_i + \frac{1}{2} \sum_{i,k=1}^{n+1} \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \theta_i \theta_k s_i s_k^T dt + \left( f_j(t, \theta(t-)) - f_j(t, \theta(t-)) \right) dQ \right.

+ \left. \left( f_j(t, \theta(t-)) - f_j(t, \theta(t)) - \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} \theta_i(t) a_i(t) \right) \Lambda(t) dt \right)
\] (3.2)

where

\[
\mu_j f_j = \frac{\partial f_j}{\partial t} dt + \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} \theta_i m_i dt + \frac{1}{2} \sum_{i,k=1}^{n+1} \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \theta_i \theta_k s_i s_k^T \left( f_j(t, \theta(t-)) - f_j(t, \theta(t)) - \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} \theta_i(t) a_i(t) \right) \Lambda(t),
\] (3.3)

\[
\sigma_{ij} f_j = \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} \theta_i s_i,
\]

\[
\alpha_j f_j = f_j(t, \theta(t-)) - f_j(t, \theta(t-)).
\]
Construct an instantaneously riskless portfolio $\Pi$ by means of $n + 2$ traded securities

$$\Pi = \sum_{j=1}^{n+2} x_j f_j,$$

Here $x_j$ is the amount of the $j$th security in the portfolio, we try to choose proper $x_j$ that are not all zero, such that the stochastic components $dz_l$ and $dQ$ are eliminated. From equation (3.2), this means that

$$\sum_{j=1}^{n+2} x_j \sigma_j f_j = 0, \quad l = 1, \ldots, n,$$
$$\sum_{j=1}^{n+2} x_j \alpha_j f_j = 0,$$  \hspace{1cm} (3.4)

then

$$d\Pi = \sum_{j=1}^{n+1} x_j \mu_j f_j dt,$$  \hspace{1cm} (3.5)

and the assumption of no arbitrage opportunities tell us

$$d\Pi = r\Pi dt,$$  \hspace{1cm} (3.6)

and then

$$\sum_{j=1}^{n+1} x_j f_j (\mu_j - r) = 0.$$ \hspace{1cm} (3.7)

And since equations (3.4) are consistent with simultaneous equations (3.4) and (3.7), the coefficient matrices of these two systems of equations have the same rank. Then there is an array of $\zeta_l, \quad l = 1, \ldots, n$ and $\zeta$ [Zhao], such that

$$f_j (\mu_j - r) = \sum_{l=1}^{n} \zeta_l \sigma_j f_j + \zeta \alpha_j f_j,$$ \hspace{1cm} (3.8)

i.e.,

$$\mu_j - r = \sum_{l=1}^{n} \zeta_l \sigma_j + \zeta \alpha_j.$$ \hspace{1cm} (3.9)

The array of $\zeta_l, \quad l = 1, \ldots, n$ and $\zeta$ can be thought as the market prices of risk produced by $z_l$ and $Q$, corresponding to $\lambda_l$ in sections 1 & 2. Here we should emphasize that $\zeta$ stands for the market price of risk produced by the jump process $Q$. With consideration of (3.3) and dropping subscripts $j$, we get the general pricing equation of derivatives whose underlying state variables exhibit jumps.
According to Merton [1976, 1994], the jumps represent “pure” non-systematic risk. When the jumps occur, the risk of jump will be completely diversifiable [Cox, 1985]. Thus, the investors require no excess return (risk premium) for the risk of jumps. And the market price of risk of jumps should be zero, i.e., $\zeta = 0$. However, we can see from the above equations that even though the jumps represent “pure” non-systematic risk, the jump component does affect the equilibrium option price. That means, one cannot “act as if” the jump component was not there (cannot ignore the jump) and compute the correct option price. And this point of view had already been emphasized by Merton [1976, 1994].

If Merton’s assumption is not true, i.e., the jump risk is systematic and non-diversifiable, and the market price of risk of jump $\zeta$ is not zero, then the term $\zeta(f(t, \theta(t-)) - f(t, \theta(t)))$ in above equation cannot be erased. The term $\zeta(f(t, \theta(t-)) - f(t, \theta(t)))$ is the key term which reflects the extra effect of non-diversifiable jump risk on valuation model to some extent, comparing with diversifiable jump risk. However, this key term has been completely ignored by former researchers of derivatives pricing.

**GENERAL PRICING EQUATION OF DERIVATIVES WITH NONLINEAR JUMPING UNDERLYING STATE VARIABLES**

Supposed that the state variables $\theta_i, \ i = 1, \ldots, n + 1$ satisfies the equations

$$d\theta_i(t) = b_i(t, \theta(t))dt + \sum_{l=1}^n c_{il}(t, \theta(t))dz_i + d_i(t, \theta(t-))dQ(t), \ i = 1, \ldots, n + 1, \quad (4.1)$$

where $b_i, c_{il}$ and $d_i$ satisfy all the conditions that Itô’s theories required, $\theta(t) \triangleq (\theta_1(t), \ldots, \theta_{n+1}(t))$. (4.1) tells us that the state variables $\theta_i, \ i = 1, \ldots, n + 1$ are nonlinear and each variable is depend on other state variables. Suppose that there are at least $n + 2$ traded derivative securities $f_j = f_j(t, \theta_1, \ldots, \theta_{n+1}) = f_j(t, \theta), \ j = 1, \ldots, n + 2$, where $\theta \triangleq (\theta_1, \ldots, \theta_{n+1})$. Applying Itô’s rule [Ikeda] to $f_j$, with the help of (4.1), and noting $\overline{\theta} \triangleq (\theta_1 + d_1(t, \theta), \ldots, \theta_{n+1} + d_{n+1}(t, \theta)), \ c_i = (c_{i1}, \ldots, c_{in}), \ c_k^T = (c_{k1}, \ldots, c_{kn})^T$, we have

$$df_j = \frac{\partial f_j}{\partial t}dt + \sum_{l=1}^{n+1} \frac{\partial f_j}{\partial \theta_l} b_l(t, \theta)dt + \sum_{l=1}^n c_{il}(t, \theta)dz_l + \frac{1}{2} \sum_{l,k=1}^{n+1} \frac{\partial^2 f_j}{\partial \theta_l \partial \theta_k} c_l c_k^T dt$$

$$+ \left\{f_j(t, \theta(t-)) - f_j(t, \theta(t-))\right\}dQ + \left\{f_j(t, \theta(t)) - f_j(t, \theta(t)-) - \sum_{l=1}^{n+1} \frac{\partial f_j}{\partial \theta_l} d_l(t, \theta)\right\}A(t)dt \quad (4.2)$$

$$\triangleq \mu_j f_jdt + \sum_{l=1}^n \sigma_{jl} f_j dz_l + \alpha_j f_j dQ(t),$$

where
\[\mu_j f_j = \frac{\partial f_j}{\partial t} + \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} b_i(t, \theta) + \frac{1}{2} \sum_{i=1}^{n+1} \sum_{k=1}^{n+1} \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} c_i \rho c_k^T + \left\{ f_j(t, \theta(t)) - f_j(t, \theta(t)) - \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} d_i(t, \theta) \right\} A(t), \]  
\eqref{4.3}

\[\sigma_j f_j = \sum_{i=1}^{n+1} \frac{\partial f_j}{\partial \theta_i} c_d(t, \theta), \]
\[\alpha_j f_j = f_j(t, \theta(t)) - f_j(t, \theta(t)). \]

By the similar procedure in section 3, we will arrive to a general pricing equation

\[\frac{\partial f}{\partial t} + \sum_{i=1}^{n+1} \frac{\partial f}{\partial \theta_i} b_i(t, \theta) - \sum_{i=1}^{n+1} c_d(t, \theta) + \frac{1}{2} \sum_{i=1}^{n+1} \sum_{k=1}^{n+1} \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} c_i \rho c_k^T (t, \theta) + \left\{ f(t, \theta(t)) - f(t, \theta(t)) - \sum_{i=1}^{n+1} \frac{\partial f}{\partial \theta_i} d_i(t, \theta) \right\} A(t) = rf + \zeta \left( f(t, \theta(t)) - f(t, \theta(t)) \right). \]

And the similar discussions and conclusion will follow. We omit the details to save the space of this paper.

**CLOSED FORM SOLUTIONS**

Suppose that there is only one state variable \( \theta \) and \( \theta = S(t) \), where \( S \) is a stock price, in a risk-neutral world, satisfying the following equation

\[dS(t) = S(t)(r dt + sdz + adQ(t)), \]  
\eqref{5.1}

similar to \( (3.1) \). And \( r \) is the risk-free rate, \( s \) and \( a \) are volatility. \( z \) is a Wiener process, whereas, the jump process is denoted by \( Q(t) = N(t) - \int_0^t \Lambda(s) ds \), \( N(t) \) is a Poisson counting process, \( \Lambda(t) \) is the intensity of jump. For simplicity, we suppose that \( r, s, a \) and \( \Lambda \) are all constants. Thus, the solution of \( (5.1) \) will be

\[S(t) = S(0)(1 + a)^{N(t)-N(0)} \exp \left\{ \left( r - a \Lambda - \frac{1}{2} s^2 \right) t + s \left[ z(t) - z(0) \right] \right\}. \]  
\eqref{5.2}

If \( N(0) = 0, z(0) = 0 \), then \( (5.2) \) becomes

\[S(t) = S(0)(1 + a)^{N(t)} \exp \left\{ \left( r - a \Lambda - \frac{1}{2} s^2 \right) t + s \cdot z(t) \right\}. \]  
\eqref{5.3}

Then the European call option price \( f \) on this stock will be

\[f(0) = E \left\{ e^{-rT} \left[ S(T) - K \right]^+ \right\} = e^{-rT} E \left[ S(T) - K \right]^+, \]  
\eqref{5.4}

where \( T \) stands for time to expiration of option, and \( K \) for strike price of option. With \( (5.3) \) taking into account, \( e^{-rT} E \left[ S(T) - K \right]^+ \) will become to
\[ e^{-rT} E \left[ S(0)(1 + a)^{N(T)} \exp \left\{ \left[ r - aA - \frac{1}{2}s^2 \right] T + s \cdot z(T) \right\} - K \right]^+ \]

\[ = e^{-rT} \sum_{n=0}^{\infty} \frac{e^{-AT}(AT)^n}{n!} E \left[ S(0)(1 + a)^n \exp \left\{ \left[ r - aA - \frac{1}{2}s^2 \right] T + s \cdot z(T) \right\} - K \right]^+ , \tag{5.5} \]

where \( z(T) \sim N(0, T) \). Then we have \( z(T) = \varepsilon \sqrt{T} \) for some \( \varepsilon \sim N(0,1) \). Using the probability density function of \( \varepsilon \), which is \( \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} \), we can calculate the expectation \( E\{\cdot\}^+ \) in (5.5) by the following formula

\[ EZ_n^+ = \int_{-\infty}^{+\infty} Z_n^+ \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{z_n \geq 0} Z_n \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz . \tag{5.6} \]

Denote \( N(d) \triangleq \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx , \quad d_{1,n} = \frac{\ln S(0)(1 + a)^n}{K} + \left( r - aA + \frac{1}{2}s^2 \right) T, \) and

\[ d_{2,n} = \frac{\ln S(0)(1 + a)^n}{K} + \left( r - aA - \frac{1}{2}s^2 \right) T \]

\[ = d_{1,n} - s\sqrt{T} . \]

It follows from (5.5) that \( Z_n \triangleq S(0)(1 + a)^n \exp \left\{ \left[ r - aA - \frac{1}{2}s^2 \right] T + s \cdot z(\sqrt{T}) \right\} - K \), and (5.6) becomes

\[ EZ_n^+ = \int_{-d_{2,n}}^{\infty} S(0)(1 + a)^n \exp \left\{ \left[ r - aA - \frac{1}{2}s^2 \right] T + s \cdot z(\sqrt{T}) \right\} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz - K \int_{-d_{2,n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz . \tag{5.7} \]

The second integral in (5.7) is easy to get and we just deduce the first one. Note that

\[ \left( r - aA - \frac{1}{2}s^2 \right) T + s \cdot z(\sqrt{T}) - \frac{z^2}{2} = (r - aA)T - \frac{1}{2}(z - s\sqrt{T})^2 , \]

and denote that

\[ y \triangleq -\left( z - s\sqrt{T} \right) , \]

then it leads to that

\[ \int_{-d_{2,n}}^{\infty} S(0)(1 + a)^n \exp \left\{ \left[ r - aA - \frac{1}{2}s^2 \right] T + s \cdot z(\sqrt{T}) \right\} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \]
Thus the result of (5.5) is as follows

$$\sum_{n=1}^{\infty} \frac{e^{-AT}}{n!} (AT)^n \left[ S(0)(1+a)^n e^{-\alpha T} N(d_{1,n}) - Ke^{-\alpha T} N(d_{2,n}) \right]$$  \hspace{1cm} (5.8)$$

Consequently, we have got a closed form solution for an option pricing model in a market with jumps

$$f(0) = \sum_{n=1}^{\infty} \frac{e^{-AT}}{n!} (AT)^n \left[ S(0)(1+a)^n e^{-\alpha T} N(d_{1,n}) - Ke^{-\alpha T} N(d_{2,n}) \right].$$  \hspace{1cm} (5.9)$$

**CONCLUSION**

We have shown the uniqueness of array $\lambda_{t}$ and have avoided some contradictions and ambiguous meaning arising. With the uniqueness of array $\lambda_{t}$, we exclude the possibility that some other array $\lambda_{t}$, which is not the market price of risk produced by $z_{t}$, but satisfies equation (1.15). Equation (3.9) shows the most general relations among the expected excess return, volatilities and the market prices of risks with non-diversifiable jump risk taking into full account. Equation (3.9) is also analogous to the capital asset pricing model, CAPM, which relates the expected excess return on stocks to their risks. It follows from Equation (3.9) that if the risk of these sudden breaks is non-diversifiable, the market will consequently pay more risk premium over the normal return rate for bearing this jump risk. And the valuation equation will be also changed. The most general and unified valuation equation of derivatives of nonlinear jumping underlying state variables is given in section 4. The non-diversifiable jump risk will produce an extra effect on the valuation equation other than diversifiable one, denoted by the term $\zeta(f(t, \theta(t-)) - f(t, \theta(t-)))$. Here $\zeta(f(t, \theta(t-)) - f(t, \theta(t-)))$ is the key term to demonstrate the effects of the non-diversifiable jump on the derivatives’ value, but this term has been completely ignored by former researchers of derivatives pricing. In some degree, this term will demonstrate the expected excess value of the derivatives to compensate the investors for the non-diversifiable jump risk exceeding diversifiable one. However, it is hard to calculate the value of $\zeta$, which stands for the market price of risk produced by the jump process $Q$ since the real effect of the future sudden and rare jump cannot be predicted, and the historic data cannot represent the future. Furthermore, the jump time is also not predictable. Although we can not calculate the value of $\zeta$, the closed form solution of option pricing can be found in a risk-neutral market with jumps under some conditions.

Since the non-diversifiable jump risk will cause tremendous effects on derivatives’ valuation by means of the key term shown above, we can say that this kind of risk is fatal to economic security. It can never be emphasized too much.
REFERENCES


