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The impact of separate processes of aggregate dividends and consumption on asset pricing with fat tails

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ABSTRACT

This paper studies a consumption-based asset pricing model in which aggregate dividends and consumption are modeled as different processes with stable shocks. The model yields mean equity returns larger to accord with the historical data than the standard framework with the assumption that dividends are identical to total consumption. This improves the potential of fat tails to explain the equity premium puzzle further. With more realistic assumption of exogenous endowment sequences, this new model also lays a broader foundation for other asset pricing models with fat tails.

KEYWORDS: Equity Premium Puzzle, Consumption-based Asset Pricing Model, Stable Paretian Distribution

INTRODUCTION

The Capital Asset Pricing Model (CAPM) attributes the large difference between the average returns on corporate equity and T-bills to a premium for bearing non-diversifiable aggregate risk. However, Mehra and Prescott find that only a small part of this difference is a premium for bearing aggregate risk (Mehra & Prescott, 1985). Therefore, they propose the equity premium puzzle. Intensive mainstream explanations of the puzzle have focused on mechanisms to increase the premium for bearing non-diversifiable risk.

The consumption-based asset pricing model proposed by Lucas is employed when Mehra and Prescott introduce the equity premium puzzle (Lucas, 1978). It is possible that the lower equity
premium derived from the Lucas model is due to an underestimation of the possibility of large shocks and hence an underestimation of related risk premium. Thus, Bidakota and McCulloch change the original exogenous endowment process implemented by Mehra and Prescott from the two-state Markov process to a first-order autoregressive process with innovations to the process drawn from the family of stable paretian distributions (Bidakota & McCulloch, 2003). They attempt to capture the ignored big shocks with such stable process and increase the equity risk premium implied by the Lucas model in order to match the observed market data.

The Bidakota and McCulloch’s stable model captures larger shocks and yields higher mean equity returns than both the Mehra and Prescott’s two-state Markov process model and the Burnside’s first-order autoregressive process model with Gaussian innovations (Burnside, 1998). But this stable model implies “only moderately greater mean equity returns”, ranging from 2.46 to 4.21, far from the historically observed average in excess of 7 percent per annum (Bidakota & McCulloch, 2003). It seems that this unencouraging result demonstrates the limitation of the ability of models with fat tails to generate realistic values of observed mean rates of return.

In accordance with the intuition of higher returns for riskier assets, the Bidakota and McCulloch’s stable model captures fat tail risks successfully but the returns are not high enough. Thus, the issue of results of the model is just as the equity premium puzzle itself as a quantitative issue. One probable reason is that Bidakota and McCulloch implement the Lucas model with ideal economic assumption that aggregate dividends equal consumption. Though they propose a first order autoregressive process with stable shocks for the log dividend growth rate, the process is estimated with the U.S. per capita consumption data following Cecchetti (Cecchetti et al, 2000). An autoregressive dividend growth process calibrated with relatively smooth consumption data leads to a process as a constant with stable shocks, while losing the volatility of dividends growth rate, hence generates more moderate return than it should be.

Though the assumption that aggregate dividends equal consumption makes sense in the ideal economic model, it is far from economic and empirical reality. Economically, pricing behavior of the claim to corporate dividends is not the same thing as the claim to the whole endowment, and stock’s claims are on corporate dividends, not consumption. Empirically, corporate dividends on average make up about 4% of consumption each year (Cecchetti et al, 1993); in addition, annual growth rates of dividends and consumption are weakly correlated in the data (Campbell, 1999; Campbell & Cochrane, 1999).

The main breakthrough of this paper is the relaxation of the above assumption with which Bidakota and McCulloch employ the Lucas model and derive solutions for asset prices and returns (Bidakota & McCulloch, 2003; Lucas, 1978). David and Veronesi find the empirical fact that the expected consumption growth rate is almost constant (David & Veronesi, 2000). And Brennan and Xia propose a representative agent model where consumption and dividends are
modeled as different processes, and the expected consumption growth rate is a constant (Brennan & Xia, 2001a). Refer to this literature, the aggregate dividends and consumption are modeled separately. In detail, the aggregate dividends growth rate is modeled as an autoregressive process with stable shocks, while the expected consumption growth rate is modeled as a constant. With more realistic assumption of exogenous endowment sequences, the new model improves the explanation ability of fat tails models for equity returns and the equity premium, thus improves its potential to solve the equity premium puzzle further.

The remainder of this paper is organized as follows. We make a summary description of the asset pricing model in the next section, the Asset Pricing Model. Then we derive the solution for asset prices when the endowment processes of aggregate dividends and consumption are modeled separately in Section III, Solution to The Model section. In Section IV, Evaluating Asset Returns, the solution for asset returns is derived. In Section V, Empirical Results of the Model, we compute the asset returns implied by the Lucas model. Then we compare our solution to that obtained by Bidarkota and McCulloch (2003) under the assumption that dividends equal consumption. We conclude in the last section.

THE ASSET PRICING MODEL

Bidarkota and McCulloch introduce the first-order Ruler condition in a single good Lucas economy with a representative agent and a single asset that pays exogenous dividends of non-storable consumption goods (Bidarkota & McCulloch, 2003; Lucas, 1978) as

\[
P_t U'(C_t) = \theta E_t U'(C_{t+1})[P_{t+1} + D_{t+1}],
\]

where \(P_t\) is the real price of the single asset in terms of the consumption good, \(U'(C)\) is the marginal utility of consumption \(C\) for the representative agent, \(\theta\) is a subjective discount factor, assumed non-stochastic and constant, \(D\) is the dividend from the single productive unit, and \(E_t\) is the mathematical expectation, conditioned on information available at time \(t\).

Assume a constant relative risk aversion (CRRA) utility function: \(U(C) = (1-\gamma)^{-1}C^{1-\gamma}, \quad \gamma \geq 0\).

Without the assumption that consumption simply equals dividends in the model, thus \(C \neq D\) every period, for separate aggregate dividends and consumption, Equation (1) reduces to

\[
P_t C_t^{-\gamma} = E_t \theta C_{t+1}^{-\gamma}[P_{t+1} + D_{t+1}],
\]

After rearranging, Equation (2) yields

\[
P_t = E_t \theta \left( \frac{C_{t+1}}{C_t} \right)^\gamma [P_{t+1} + D_{t+1}].
\]
As Bidarkota and McCulloch (2003) process, let \( v_i \) be the price-dividend ratio, that is \( v_i = P_i / D_i \). Then Equation (3) can be reformed in terms of \( v_i \) as

\[
 v_i = E_t \left( \frac{C_{t+1}}{C_t} \right)^\gamma \left( \frac{D_{t+1}}{D_t} \right) [v_{t+1} + 1].
\]

(4)

Let \( x_i = \ln(D_i / D_{t-1}) \) be the log dividend growth rate, and \( y_i = \ln(C_i / C_{t-1}) \) be the log consumption growth rate. Then we can rewrite Equation (4) in terms of \( x_i \) and \( y_i \) as

\[
 v_i = E_t \theta \exp[-\gamma y_{t+1} + x_{t+1}] (v_{t+1} + 1).
\]

(5)

In Veronesi (2000), consumption and dividends are modeled as the same process, and the expected consumption growth rate is time varying. Bidarkota and McCulloch (2003) do the same and assume that consumption simply equals dividends, i.e. \( C = D \) every period. Hence they rewrite Equation (4) in terms of \( x_i \), as \( v_i = E_t \theta \exp[(1-\gamma) x_{t+1}] (v_{t+1} + 1) \). Nonetheless, later in their empirical assessment section, they estimate their endowment process with observed data on U.S. per capita consumption data following Cecchetti et al (Cecchetti et al, 2000). This coincides with their assumption that consumption and dividends are identical. In fact, they rewrite Equation (4) in terms of \( y_i \) as

\[
 v_i = E_t \theta \exp[(1-\gamma) y_{t+1}] (v_{t+1} + 1).
\]

(6)

**SOLUTION TO THE MODEL**

In this section, we discuss reasons for separate processes of aggregate dividends and consumption. Both the log dividend growth rate \( x_i \) and the log consumption growth rate \( y_i \) are specified as different processes. Then an analytical solution for \( v_i \) is obtained.

**Discussion of the Endowment Processes Specification**

There are two special cases for the expected consumption growth rate. One is that the consumption growth rate is a constant, such as the assumption made by David and Veronesi (David & Veronesi, 2000), and Brennan and Xia (Brennan & Xia, 2001). The other is that the
expected consumption growth rate is time varying with the assumption that consumption equals aggregate dividends. And this ideal assumption is exploited by Mehra and Prescott (Mehra & Prescott, 1985), Veronesi (Veronesi, 2000) and Bidarkota and McCulloch (Bidarkota & McCulloch, 2003).

Though the identical consumption and dividends assumption simplifies the implementation of Lucas model (Lucas, 1978), it is inconsistent with empirical facts. Consumption growth is considerably smoother than dividend growth. David and Veronesi find the expected growth rate of consumption is almost constant (David & Veronesi, 2000). Even Bidarkota and McCulloch estimate the process of dividends with consumption data and get constant growth rate as well (Bidarkota & McCulloch, 2003). In addition, annual growth rates of dividends and consumption are weakly correlated in the data (Campbell, 1999; Campbell & Cochrane, 1999).

Economically speaking, the identical consumption and dividends assumption is also in contrast with the realistic situation. Pricing behavior of the claim to corporate dividends is not the same thing as the claim to the entire endowment, because stock’s claims are on corporate dividends, not consumption. Therefore, separation processes of consumption and dividends are more realistic, and the expected consumption growth rate should be a constant while the dividends growth rate should be varying.

According to both the empirical and economic reasons discussed above, we assume the log dividend growth rate $x_t$ following the process as

$$x_t = (1 - \rho) \mu + \rho x_{t-1} + \varepsilon_t, \quad |\rho| < 1, \quad (7)$$

and the log consumption growth rate $y_t$ is

$$y_t = \eta + \varepsilon_t, \quad (8)$$

where $\varepsilon_t \sim iid(\alpha, \beta, c, 0)$. $S(\alpha, \beta, c, 0)$ represents a stable paretian distribution with characteristic exponent $\alpha$, skewness parameter $\beta$, scale parameter $c$, and location parameter set to zero. McCulloch defines the distribution and lists some of their properties (McCulloch, 1996).

Analytical Solution of the Price-dividend Ratio

Let $m_t = \theta \exp[-\gamma y_t + x_t]$, Equation (5) can be reduced to

$$v_t = E_t m_{t+1} | v_{t+1} + 1]. \quad (9)$$
By forward iterating on Equation (9), the solution for $v_i$ is given by
\[
v_i = \sum_{j=1}^{\infty} \left( E_i \prod_{j=1}^{i} m_{t+j} \right) + \lim_{i \to \infty} E_i \prod_{j=1}^{i} m_{t+j} v_{t+i},
\] (10)

As Bidakota and McCulloch (2003), we focus on the particular solution by imposing the transversely condition: $\lim_{i \to \infty} E_i \prod_{j=1}^{i} m_{t+j} v_{t+i} = 0$. Solutions to the asset pricing model that implies intrinsic bubble is excluded out by the above condition (Froot & Obstfeld, 1991). In this case, Equation (9) yields
\[
v_i = \sum_{j=1}^{\infty} E_i \prod_{j=1}^{i} m_{t+j},
\] (11)

**EVALUATING ASSET RETURNS**

In this section, the return on risk free asset in our separate endowment economy is derived at first. Then the return on risky assets is given by both the Bidakota and McCulloch (2003) economy and our new endowment economy. And we make the comparison between these two frameworks.

**Returns on Risk Free Assets**

In Lucas endowment economy (Lucas, 1978), the price of a risk free asset $P_t^f$ ensures one unit of the consumption good at maturity. So we have
\[
P_t^f = \theta E_i \left[ U'(C_{t+1}) \right] / U'(C_t).
\] (12)

With the constant relative risk aversion (CRRA) utility function, and the log consumption growth rate $y_t$, Equation (12) becomes
\[
P_t^f = \theta E_i \left[ \exp(-\gamma y_{t+1}) \right].
\] (13)

Under the process defined as the log consumption growth rate Equation (8), when $y_t$ has the skewness parameter $\beta = +1$, Appendix 1 shows the price of a risk free asset $P_t^f$ is yielded as
\[ P_t' = \theta \exp \left[ -\gamma \eta - (\gamma c)^\alpha \sec \left( \frac{\pi \alpha}{2} \right) \right]. \]  

Equation (14) shows that returns on the risk free asset \( R_t' \) under gross equilibrium are given by \( R_t' = \frac{1}{P_t'} \). Thus risk free returns are given by

\[ R_t' = \theta^{-1} \exp \left[ \gamma \eta + (\gamma c)^\alpha \sec \left( \frac{\pi \alpha}{2} \right) + \gamma \rho (x_t - \eta) \right]. \]  

Equation (15) as \( R_t' = \theta^{-1} \exp \left[ \gamma \eta + (\gamma c)^\alpha \sec \left( \frac{\pi \alpha}{2} \right) + \gamma \rho (x_t - \eta) \right] \). For the risk free return is the benchmark of the equity premium, their results under the unrealistic assumption affects the calculation of both the risk free return and the equity premium.

In the case where \( \gamma_t \) has the skewness parameter \( \beta = -1 \), Appendix 1 shows that \( P_t' \) is infinite. Probably, huge uncertainty with intensive shocks determines the endowment process, and infinite amount would be paid by investors who are risk averse to avoid such kind of extreme environment, thus risk free returns become zero consequently.

\[ R_t' = \exp \left[ \frac{1 + v_{t+1}}{v_t} \right] \exp [x_{t+1}] . \]  

Equation (16) shows that returns on risky assets as the definition in Bidakota and McCulloch's paper, the equilibrium gross equity returns \( R_t^e \) on assets held from period \( t \) through period \( t+1 \) are given by

\[ R_t^e = \frac{P_{t+1} + D_{t+1}}{P_t} . \]

Substituting the price-dividend ratio \( v_t = P_t / D_t \) and the log dividend growth rate \( x_t = \ln \left( \frac{D_t}{D_{t-1}} \right) \), this reduces to

\[ R_t^e = \left( \frac{1 + v_{t+1}}{v_t} \right) \exp [x_{t+1}] . \]  

Equation (16) shows that returns on risky assets.
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The Impact of Separate Processes on Asset Pricing

Since $v_i$ is a function of $x_i$ and $y_i$ as Equation (11) with $m_i = \theta \exp[-\gamma y_i + x_i]$, the mean of the implied equity returns is hard to derive. We can expand $v_i$ and gain insight how complex it is.

$$v_i = \sum_{j=1}^{\infty} \left( E_i \prod_{j=1}^{j} m_{t+j} \right)$$

$$v_i = E(m_{t+1}) + E(m_{t+1}m_{t+2}) + E(m_{t+1}m_{t+2}m_{t+3}) + \ldots + E(m_{t+1}m_{t+2}m_{t+3} \ldots m_{t+n}). \quad (17)$$

Conveniently, Bidakota and McCulloch regard $x_i$ as $y_i$ implied by the assumption that aggregate dividends equal consumption. Therefore, the items in Equation (17) such as $E(m_{t+1})$, $E(m_{t+1}m_{t+2})$, and so on deduce to constants. Consequently the summation of these constants becomes the constant $v$ which is independent of time $t$.

This coincides with the explanation in Bidakota and McCulloch’s paper that in the case of a random walk for dividend growth rates, $p = 0$ and $v_i$ reduces to constant (Bidakota & McCulloch, 2003). In such simpler case, they derive an exact analytical expression for $E(R^c_t)$ as

$$E(R^c_t) = \left( \frac{1+v}{v} \right) \exp \left[ \mu - c^\alpha \sec \left( \frac{\pi \alpha}{2} \right) \right]. \quad (18)$$

And they employ Equation (18) to calculate returns on risky assets. In fact, their solutions consist of two major problems. First, they replace the dividend growth rate process by the consumption growth rate process, under the relative unrealistic assumption that two processes are identical. Second, they estimate the dividend process with consumption data and get parameters of stable innovation $\epsilon_i \sim iid(S(\alpha, \beta, c, 0))$.

In order to avoid these problems, under the realistic assumption of separate aggregate dividends and consumption processes as $x_i$ and $y_i$, we get the solution to Equation (16) in Appendix 2.

With steps given in Appendix 2, the numerical solution can be obtained and compared with the result of Equation (18) employed by Bidakota and McCulloch in the next section.

**EMPIRICAL RESULTS OF THE MODEL**
From the analytical solution given in Appendix 2, we have

\[ E[R_t^+] = E \left[ \frac{1}{v_t^2} E \left[ \exp(x_{t+1}) + v_{t+1} \exp(x_{t+1}) \right] \right]. \]  \hspace{1cm} (19.a)

And the above equation can be viewed as

\[ E[f(x_t)] = \int_{-\infty}^{\infty} f(x_t) p_{x_t} dx_t, \]  \hspace{1cm} (19.b)

where \( p_{x_t} \) is the probability distribution function (pdf) of \( x_t \). Then we employ Simpson’s rule to compute the integral in Equation (19.b). And the pdf \( p_{x_t} \) can be obtained by using Zolotarev’s proper integral representations (Zolotarev, 1986). Also we can implement the computational algorithm developed by J.P. Nolan.

For convenience of comparison between results of our model and the Bidakota and McCulloch’s model, we use the same parameters estimated to determine whether our model yields higher implied mean equity returns than theirs. They estimate their endowment process with the U.S. per capita consumption data extending from 1890 through 1987, following Cecchetti et al (Cecchetti et al, 2000). And in Table 1 of Bidakota and McCulloch’s paper (Bidakota & McCulloch, 2003), assuming \( x_t = \mu + \varepsilon_t, \varepsilon_t \sim iid(\alpha, \beta, c, 0) \), when \( \beta = -1 \), they get \( \alpha = 1.8703, c = 0.0245, \mu = 0.0223 \). Recall that our endowment processes are modeled as Equation (7) and Equation (8) as

\[
\begin{cases}
    x_t = (1 - \rho) \mu + \rho x_{t-1} + \varepsilon_t, & |\rho| < 1, \\
    y_t = \eta + \varepsilon_t, & \varepsilon_t \sim iid(\alpha, \beta, c, 0).
\end{cases}
\]

Input parameters above while assuming \( \eta = \mu \), we have our results with different \( \rho \) s in Table 1.

**Table 1. Model-implied mean equity returns**

Table 1 provides mean equity returns of the model with separate processes of aggregate dividends and consumption. \( \theta \) is the discount factor in the Lucas (1978) model. \( \gamma \) is the risk averse parameter in the CRRA utility function and \( \gamma > 0 \). \( R^\gamma \) is the model-implied mean equity returns. And \( \rho \) is the autoregressive parameter which satisfies \( |\rho| < 1 \).
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The Impact of Separate Processes on Asset Pricing

From Table 1 we can see that, as \( r \) become larger, model-implied mean equity returns \( R^e \) get higher. Presumably, larger autoregressive parameters would amplify the impact of shocks on the dividends growth rate \( x \), thus result in higher mean equity returns. Interestingly, while \( r \) equals zero, our model reduces to the standard stable model with the assumption that aggregate dividends and consumption are identical. And our model yields the same results as the Bidakota and McCulloch model. In the same words, their stable model emerges as a special case of our model.

Table 2. Comparison between different model-implied mean equity returns

Table 2 provides mean equity returns of three different models. From left to right, results of our new model with different values of \( \rho \) are in the third and fourth columns. Results of the Bidakota and McCulloch (2003) model are in the fifth column. And results of the Burnside (1998) model are in the sixth column. The meaning of parameters can be referred back to Table 1.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \gamma )</th>
<th>( 100(R^e - 1) )</th>
<th>( 100(R^e - 1) )</th>
<th>( 100(R^e - 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = 0 )</td>
<td>( \rho = 0.2 )</td>
<td>( \rho = 0.4 )</td>
<td>( \rho = 0.6 )</td>
<td>Stable model</td>
</tr>
<tr>
<td>0.98</td>
<td>0.60</td>
<td>3.50</td>
<td>3.52</td>
<td>3.56</td>
</tr>
<tr>
<td>0.98</td>
<td>0.75</td>
<td>3.86</td>
<td>3.88</td>
<td>3.93</td>
</tr>
<tr>
<td>0.98</td>
<td>0.90</td>
<td>4.21</td>
<td>4.24</td>
<td>4.31</td>
</tr>
<tr>
<td>0.99</td>
<td>0.60</td>
<td>2.46</td>
<td>2.47</td>
<td>2.53</td>
</tr>
<tr>
<td>0.99</td>
<td>0.75</td>
<td>2.81</td>
<td>2.83</td>
<td>2.89</td>
</tr>
</tbody>
</table>

It is obvious from Table 2 that for all combinations of the independent parameters employed, our new model with separate processes of dividends and consumption generates higher mean equity returns than the corresponding standard stable model in Bidakota and McCulloch’s paper which assumes aggregate dividends equals consumption. This is in accordance with the intuition of higher returns for riskier assets. And the results of our new model are closer to the historically observed average at 7 percent per annum. Notice that we here use the same parameters in
Bidakota and McCulloch’s paper which are estimated by the consumption data for the convenience of comparisons. Results of our models would be higher when parameters estimated by dividends data are implemented.

CONCLUSION

We study a consumption-based asset pricing model in which aggregate dividends and consumption are modeled as different processes with stable shocks. Then we derive solutions to the model. The new model introduces separate processes with stable shocks of aggregate dividends and consumption to the Lucas (1978) model. By relaxing the assumption that aggregate dividends equal consumption, the equity returns implied by the model are greater than derived in the Bidakota and McCulloch (2003) stable model. This new framework with such more realistic assumption generates mean equity rates of return that are larger to accord with historical data, thus enhances the ability of the fat tails as a risk factor that can explain the equity premium puzzle.

With more realistic assumption of exogenous endowment sequences, this new model also lays a broader foundation for other asset pricing models with fat tails. After Bidakota and McCulloch introduce fat tails to the Lucas model with reference to Burnside (Burnside, 1998), Bidarkota and Dupoyet introduce the habit formation to the standard stable model (Bidarkota and Dupoyet, 2007). Later Bidarkota, Dupoyet and McCulloch add the incomplete information to the standard stable model as well (Bidarkota et al, 2009). While their efforts for enhancing the ability of the fat tails model to explain the equity premium are successful to some extent, they all model both aggregate dividends and consumption with the same process and employ the consumption process only. Based on their efforts, it is probable that results of our new model with the habit formation and incomplete information would perform even better.

APPENDIX 1. Derivation of the Risk Free Asset Prices

From Section IV subsection Returns on The Risk Free Asset, the price of the risk free asset is

\[ P_t^r = \theta E_t \left( \frac{U'(C_{t+1})}{U'(C_t)} \right) \]

With the constant relative risk aversion (CRRA) utility function in Section II, this deduces to

\[ P_t^r = \theta E_t \left( \frac{C_{t+1}}{C_t} \right)^\gamma. \]  \hspace{1cm} (A1.1)

Employing the log consumption growth rate \( \gamma \), Equation (A1.1) becomes

\[ P_t^r = \theta E_t \left[ \exp(-\gamma \gamma_{t+1}) \right]. \]
Substituting the definition of the log consumption growth rate process Equation (8), we get

\[ P_t^f = \theta E_t \left[ \exp \left( -\gamma \eta - \gamma e_{t+1} \right) \right]. \] (A1.2)

Since in Equation (8) the stable shocks means \( e_t \sim \text{iid } S(\alpha, \beta, c, 0) \), according to the stable distribution definition of transformation in McCulloch (1996) that \( aX : S(\alpha, \text{sign}(\gamma)\beta, |\gamma|c, a\delta) \), we have

\[ -\gamma e_t : \text{iid } S(\alpha, \text{sign}(\gamma)\beta, |\gamma|c, 0). \]

Since \( \gamma > 0 \),

\[ -\gamma e_t : \text{iid } S(\alpha, -\beta, \gamma c, 0). \]

Also according to the stable distribution feature in McCulloch (1996) that when \( \beta = -1 \),

\[ \ln E e^x = \delta - c^\alpha \sec \left( \pi \alpha / 2 \right), \]

we have case 1 and case 2.

Case 1: \( \beta = 1 \),

\[ E_t \left[ \exp \left( -\gamma e_{t+1} \right) \right] = \exp \left[ -\gamma c^\alpha \sec \left( \pi \alpha / 2 \right) \right]. \]

Therefore, substituting into Equation (A1.2), we have

\[ P_t^f = \theta \exp \left[ -\gamma \eta - \gamma c^\alpha \sec \left( \pi \alpha / 2 \right) \right]. \]

This is Equation (14) in the text.

Case 2: \( \beta = -1 \).

In McCulloch (1996), when \( \beta = -1 \), we have

\[ \ln E e^x = \delta - c^\alpha \sec \left( \pi \alpha / 2 \right), \]

which implies

\[ E_t \left[ \exp \left( -\gamma e_{t+1} \right) \right] = \infty. \]

Equation (A1.2) then implies that \( P_t^f = \infty \).

**APPENDIX 2. Derivation of Returns on the Risky Asset**

From the subsection Returns on Risky Assets in Section IV, the equilibrium gross equity returns \( R_t^e \) on assets held from period \( t \) through period \( t + 1 \) are given by Equation (16) as
\[ R_t = \left( \frac{1 + v_{t+1}}{v_t} \right) \exp[x_{t+1}] \]

And \( v_t \) is defined as

\[ v_t = \sum_{m=1}^{\infty} \left( E_t \prod_{j=1}^{m} m_{t+j} \right) \]

where \( m_t = \theta \exp[-\gamma y_t + x_t] \). Then we derive the population mean of the implied equity returns \( E[R_t] \).

With the notion that given information set including both \( x_t \) and \( y_t \) as \( F_t = \sigma \{ x_t, y_t \} \), i.e. the \( \sigma \)-algebra is generated by \( x_t \) and \( y_t \), we have \( E_t[X] = E[X|F_t] \). Then we obtain:

\[
\begin{align*}
E[R_t^+] &= E \left( \left( \frac{1 + v_{t+1}}{v_t} \right) \exp[x_{t+1}] \right) \\
E[R_t^-] &= E \left[ E_t \left( \left( \frac{1 + v_{t+1}}{v_t} \right) \exp[x_{t+1}] \right) \right] \\
E[R_t^0] &= E \left[ \frac{1}{v_t} \left( \exp(x_{t+1}) + v_{t+1} \exp(x_{t+1}) \right) \right].
\end{align*}
\]

(A2.1)

According to the structure of Equation (A2.1), we solve it in three steps.

Step 1: for the algebra term \( v_t \),

For convenience, we reform \( x_{t+1} \) as \( x_{t+1} = A + Bx_{t+n-1} + \epsilon_{t+n} \), thus

\[
\begin{align*}
x_{t+n} &= A + Bx_{t+n-1} + \epsilon_{t+n} \\
&= A + B \left( A + Bx_{t+n-2} + \epsilon_{t+n-1} \right) + \epsilon_{t+n} \\
&= A \left( 1 + B + B^2 + \cdots + B^{n-1} \right) + \epsilon_{t+n} + B\epsilon_{t+n-1} + \cdots + B^{n-1} \epsilon_{t+1} + B^n \epsilon_t.
\end{align*}
\]

Then we can sum \( x_t \) as
\[
x_{r+1} + x_{r+2} + \cdots + x_{r+n} \\
= A\left(1 + (1 + B) + (1 + B + B^2) + \cdots + (1 + B + B^2 + \cdots + B^{n-1})\right) \\
+ \epsilon_{r+n} + (1 + B)\epsilon_{r+n-1} + (1 + B + B^2)\epsilon_{r+n-2} + \cdots + (1 + B + B^2 + \cdots + B^{n-1})\epsilon_{r+1} \\
+ (B + B^2 + \cdots + B^{n-1})x_r,
\]

and have
\[
E_r\left[m_{r+1}m_{r+2}\cdots m_{r+n}\right] \\
= \theta^n \exp[-\eta B + A\left(1 + (1 + B) + (1 + B + B^2) + \cdots + (1 + B + B^2 + \cdots + B^{n-1})\right)] \\
E\left[\exp\left((1 - \gamma)\epsilon_{r+n}\right)\right]E\left[\exp\left((1 + B - \gamma)\epsilon_{r+n-1}\right)\right] \\
E\left[\exp\left((1 + B + B^2 - \gamma)\epsilon_{r+n-2}\right)\right]\cdots E\left[\exp\left((1 + B + B^2 + \cdots + B^{n-1} - \gamma)\epsilon_{r+1}\right)\right] \\
\exp\left((B + B^2 + \cdots + B^n)x_r\right).
\]

(A2.2)

Then, with Equation (17), we can get \( v_r \).

Step 2: for the algebra term \( v_{r+1} \exp(x_{r+1}) \),

According to Equation (A2.2), we have
\[
E_{r+1}\left[m_{r+1}m_{r+2}\cdots m_{r+n}\right]\exp(x_{r+1}) \\
= \theta^n \exp[-\eta B + A\left(1 + (1 + B) + (1 + B + B^2) + \cdots + (1 + B + B^2 + \cdots + B^{n-1})\right)] \\
E\left[\exp\left((1 - \gamma)\epsilon_{r+n}\right)\right]E\left[\exp\left((1 + B - \gamma)\epsilon_{r+n-1}\right)\right] \\
E\left[\exp\left((1 + B + B^2 - \gamma)\epsilon_{r+n-2}\right)\right]\cdots E\left[\exp\left((1 + B + B^2 + \cdots + B^{n-1} - \gamma)\epsilon_{r+1}\right)\right] \\
\exp\left((B + B^2 + \cdots + B^n)x_{r+1}\right).
\]

(A2.3)

Then we obtain \( v_{r+1} \exp(x_{r+1}) \) with Equation (17) as well.

Step 3: for the algebra term \( E_r\left[v_{r+1} \exp(x_{r+1})\right] \),

As Step 2 and Step 3, with Equation (A2.3), we get
\[ E \left[ E_{t+1} \left[ m_{t+1} m_{t+2} \cdots m_{t+n+1} \right] \exp\left( x_{t+1} \right) \right] \]

\[ = \theta^n \exp\left[ -n \gamma \eta + A \left( 1 + (1 + B) + (1 + B + B^2) + \cdots + (1 + B + B^2 + \cdots + B^n) \right) \right] \]

\[ E \left[ \exp\left( (1 - \gamma) e_{t+n} \right) \right] E \left[ \exp\left( (1 + B - \gamma) e_{t+n} \right) \right] \]

\[ E \left[ \exp\left( (1 + B + B^2 - \gamma) e_{t+n-1} \right) \right] \cdots E \left[ \exp\left( (1 + B + B^2 + \cdots + B^{n-1} - \gamma) e_{t+2} \right) \right] \]

\[ E \left[ \exp\left( (1 + B + B^2 + \cdots + B^n) e_{t+1} \right) \right] \exp\left( (B + B^2 + \cdots + B^{n+1}) x_t \right). \]

By Equation (17), the algebra term \( E_t \left[ v_{t+1} \exp\left( x_{t+1} \right) \right] \) can be obtained.

Finally, according to the feature of stable distribution, as \( \varepsilon_i \sim iid (\alpha, \beta, c, \delta) \), we get

\[ E \left[ \exp(k \varepsilon_i) \right] = \exp \left[ k \delta - k^\alpha \sec\left( \frac{\pi \alpha}{2} \right) \right]. \]  

And with the results of algebra terms from Step 1 to Step 3, we can get the mean of the implied equity returns \( E \left[ R_{t+1}^e \right] \) with Equation (A2.1).

REFERENCES


