ABSTRACT

In the field of stochastic consumer finance market with jumps, we study portfolio and wealth processes decided by initial endowment or by initial and terminal wealth, as well as contingent claims and option valuation decided by terminal wealth. Comparing “deflator” with “discount factor”, we give two kinds of proofs for each important theorem by stochastic analysis method. And we give the necessary sufficient conditions for the hedging strategies to duplicate completely the derivative securities contingent claims in markets, under the conditions of deterministic coefficients, using stochastic analysis and partial differential-difference equations.

KEYWORDS: Portfolio management, Initial endowment, Terminal wealth, Contingent claims, Valuation

INTRODUCTION

According to Merton’s point of view [33, 34], the total change in the stock price is posited to be the composition of two types of changes: (1) The normal vibrations in price, for example, due to a temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook, or other new information that causes marginal changes in the stock’s value. In essence, the impact of such information per unit time on the stock price is to produce a marginal change in the price (almost certainly). This component is modeled by a standard geometric Brownian motion with a constant variance per unit time and it has a continuous sample path. In this situation, there is ample scope for the famous Black-Scholes formula. (2) The abnormal vibrations in price are due to the arrival of important new information about the stock that has more than a marginal effect on price, for example, due to the Sept. 11 attacks, which caused disasters in the world as well as in the financial markets. Usually, such information will be specific to the firm or possibly its industry. It is reasonable to expect that there will be active times in the stock when such information arrives and quiet times when it does not arrive although the active and quiet times are random. According to its very nature, important information arrives only at discrete points of time. This component is modeled by a jump process reflecting the non-marginal impact of the information (This jump process can also be considered as a model reflecting the sudden breaks). And on this occasion of abnormal vibrations, the Black-Scholes formula is not valid, even in the continuous limit, because the stock price dynamics cannot be represented by a stochastic process with a continuous sample path. We must use stochastic differential equations with jumps to study those problems on portfolio and wealth processes decided by initial endowment or by initial and terminal wealth, as well as problems on valuation of contingent claims and option decided by terminal wealth. In this
paper, comparing "deflator" with "discount factor", we show two kinds of proofs for some important theorems on investment decisions when choosing portfolio and wealth processes, using stochastic analysis methodology. And we give the necessary sufficient conditions for the hedging strategies to duplicate completely the derivative securities contingent claims in the markets, under the conditions of deterministic coefficients, using stochastic analysis and partial differential-difference equations, cf. [2, 21, 22, 23, 24, 25, 33, 34, and 42].

1. The financial market model

Let us first introduce a point process \( \{ n_t : n \geq 1 \} \), where \( t_n \) is the time of the \( n \) th jump. We denote

\[
N(t) = \sup \{ n : t_n \leq t \} = \# \{ t_n : t_n \leq t \}
\]

(1.1)
as the number of random jumps to the market by time \( t \). \( N \) is a counting process associated with \( \{ t_n : n \geq 1 \} \), it will represent both the jump process, as well as the term in (1.1).

Let’s consider a financial market subject to both diffusive uncertainty and jump uncertainty. Uncertainty comes from an \( \mathbb{R}^d \)-valued Brownian motion \( W(t) = (W_1(t), \ldots, W_d(t))^T \), and a one-dimensional jump process \( N(t) \) denoted above. \( W(t) \) is defined on a probability space \( (\Omega, \mathcal{F}, P) \) and \( N(t) \) on \( (\Omega^N, \mathcal{F}^N, P^N) \).

Let \( (\Omega, \mathcal{F}, P) \) be the product probability space, i.e., \( \Omega = \Omega^W \times \Omega^N, \mathcal{F} = \mathcal{F}^W \otimes \mathcal{F}^N \), and \( P = P^W \otimes P^N \). There are \( d + 1 \) sources of uncertainty present altogether. We assume that there is a fixed time horizon \( 0 \leq T < \infty \).

**Assumption 1.** The jump process \( N(t) \) have a \((P, \mathcal{F})\)-stochastic intensity \( \lambda(t) \), in other words, \( \lambda(t) \) is the rate of the jump process at time \( t \). The process \( \lambda(t) \) is \( \{ \mathcal{F}_t \} \)-predictable, positive and uniformly bounded away from 0 on \( [0, T] \).

For more details on point process, see Bremaud [7].

Because of that \( N(t) \geq N(s), \ t \geq s \geq 0 \), we have that

\[
E(N(t)|\mathcal{F}_s) \geq E(N(s)|\mathcal{F}_s) = N(s),
\]
this means that \( N \) is a submartingale.

The point process \( \{ t_n : n \geq 1 \} \) is also a sequence of stopping times. Indeed,

\[
\{ \omega : N(t) < n \} = \{ \omega : t_n > t \},
\]
and \( N(t) \) is \( \mathcal{F}_t \)-adapted, thus we have that \( \{ \omega : N(t) < n \} \in \mathcal{F}_t \), this implies

\[
\{ \omega : t_n > t \} \in \mathcal{F}_t,
\]
i.e., \( t_n \) is a stopping time.
In the market we consider, there are \( d + 2 \) securities (assets) being traded continuously. The first one is riskless, called a bond, with price \( P_0(t) \) given by
\[
dP_0(t) = P_0(t)r(t)dt, \quad P_0(0) = 1.
\]
The other \( d + 1 \) securities are risky assets, call stocks, subject to the uncertainty in the market. The prices are modeled by the linear stochastic equations
\[
dP_i(t) = P_i(t)\left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t) + \rho_i(t)dQ(t)\right] = P_i(t)\left[b_i(t)dt + \sigma_i(t)dW(t) + \rho_i(t)dQ(t)\right], \quad P_i(0) = p_i,
\]
for \( i = 1, \ldots, d + 1 \), where
\[
Q(t) = N(t) - \int_0^t \lambda(s)ds,
\]
is a \( P \)-martingales and represents the contribution of the jumps to the security returns. \( r(t) \) is the instantaneous rate of interest. \( b(t) = (b_1(t), \ldots, b_{d+1}(t))^T \) is the vector of the instantaneous appreciation rates on the stocks. \( \sigma(t) \triangleq (\sigma_{ij}) \) is a \((d+1) \times d\) volatility matrix process and \( \rho \triangleq (\rho_i) \) is a \((d+1) \times 1\) one. Then \( \sigma(t) \triangleq [\sigma(t)] \) is a \((d+1) \times (d+1)\) volatility matrix process. \( \sigma_i \) denotes the \( i \)th row vector of \( \sigma \).

Assumption 2. \( r(t), b(t), \sigma(t) \) are predictable with respect to \( \{\mathcal{F}_t\} \), and are bounded uniformly in \((t, \omega) \in [0,T] \times \Omega\). Furthermore, \( 1 + \rho_i(t) > 0 \) for all \( i \) and \( t \in [0, T] \), to ensure limited liability of the stock. And the covariance matrix process \( a(t) \triangleq \sigma(t)\sigma^T(t) \) is assumed to be strongly nondegenerate [5,21].

Remark. The assumption \( 1 + \rho_i(t) > 0 \) is also essential in mathematics, we will see this later.

Note that the sizes of the jumps in the security returns are random, with the randomness coming from the process \( \rho \). And the jump process is not predictable (the jump process is right continuous and left limited, abbreviated RCLL). However, the effect of a jump is predictable, since \( \rho \) is predictable. This means that, at time \( t^- \), the effect of a possible jump at \( t \) is known.

From martingale theory, the processes \( W \) and \( Q \) of (1.3) are actually \( P \)-martingales, and the price process \( P_i \) of stock \( i \), is a semimartingale with drift rate \( b_i(t) \), \( i = 1, \ldots, d + 1 \).

Define the discount factor as
\[
\beta(t) \triangleq \frac{1}{P_0(t)} = \exp\left(-\int_0^t r(s)ds\right).
\]

Now we envision a small investor who starts with some initial endowment \( x \geq 0 \) and invests
it in the \(d+2\) securities described previously. Let \(\eta_i(t)\) denote the number of shares of security \(i\) held by the investor at time \(t\). Then \(X(0) \equiv x = \sum_{i=0}^{d+1} \eta_i(0)p_i, p_0 = 1\), and the investor's wealth at time \(t\) is

\[
X(t) = \sum_{i=0}^{d+1} \eta_i(t)p_i(t)
\]  

(1.6)

If transaction of shares takes place at discrete time points, for example, at \(\tau\) and \(\tau + h\), and \(\tau \neq t_n, \tau + h \neq t_n\) (i.e., transaction never takes place at time of jump), and there is no infusion or withdrawal of funds (i.e., taking a self-financing strategy), then it is obvious that \(\eta_i(t)\) = constant for \(t \in (\tau, \tau + h)\). Assume that \(\eta_i(\tau + 0) = \eta_i(\tau)\), then \(\eta_i\) is right-continuous and with left-limit at transaction time point. We should emphasize the dynamic property of \(X\) at time \(\tau\) and \(\tau + h\).

We claim that \(X(\tau - 0) = X(\tau + 0)\). Indeed,

\[
X(\tau - 0) = \sum_{i=0}^{d+1} \eta_i(\tau - 0)p_i(\tau), \quad X(\tau + 0) = \sum_{i=0}^{d+1} \eta_i(\tau + 0)p_i(\tau),
\]

and \(p_i(\tau - 0) = p_i(\tau + 0) = p_i(\tau)\) since \(p_i\) is continuous at \(\tau\) (for \(\tau \neq t_n\)). Suppose that at time \(\tau\), the investor sells (shorts) a set \(I \subset \{0, 1, \cdots, d+1\}\) of securities, and buys (longs) simultaneously \(\bar{I} \triangleq \{0, 1, \cdots, d+1\} - I\), i.e., the rest securities traded. Since the investor takes a self-financing strategy, we have the following equation

\[
\sum_{i \in I}[\eta_i(\tau - 0) - \eta_i(\tau + 0)]p_i(\tau) = \sum_{i \notin \bar{I}}[\eta_i(\tau + 0) - \eta_i(\tau - 0)]p_i(\tau), \quad \text{and then}
\]

\[
\sum_{i=0}^{d+1} \eta_i(\tau - 0)p_i(\tau) = \sum_{i=0}^{d+1} \eta_i(\tau + 0)p_i(\tau), \quad \text{i.e.,} \quad X(\tau - 0) = X(\tau + 0).
\]

Thus, define \(X(\tau) \triangleq X(\tau - 0) = X(\tau + 0)\). And similarly,

\[
X(\tau + h) \triangleq X(\tau + h - 0) = X(\tau + h + 0).
\]

So

\[
X(\tau + h) - X(\tau) = X(\tau + h - 0) - X(\tau + 0) = \sum_{i=0}^{d+1} \eta_i(\tau + h - 0)p_i(\tau + h) - \sum_{i=0}^{d+1} \eta_i(\tau + 0)p_i(\tau)
\]

\[
= \sum_{i=0}^{d+1} \eta_i(\tau)p_i(\tau + h) - \sum_{i=0}^{d+1} \eta_i(\tau)p_i(\tau) = \sum_{i=0}^{d+1} \eta_i(\tau)[p_i(\tau + h) - P_i(\tau)].
\]  

(1.7)

On the other hand, if the investor chooses at time \(\tau + h\) to consume an amount \(hC(\tau + h)\) and reduce the wealth accordingly, then (1.7) should be replaced by

\[
X(\tau + h) - X(\tau) = \sum_{i=0}^{d+1} \eta_i(\tau)[P_i(\tau + h) - P_i(\tau)] - hC(\tau + h).
\]  

(1.8)
The continuous-time analogue of (1.8) is
\[ dX(t) = \sum_{i=0}^{d} \eta_i(t) dP_i(t) - C(t) dt. \]

Taking (1.2), (1.3), (1.6) into account and denoting the amount invested in the \( i \)th stock by \( \pi_i(t) \), we obtain
\[ dX(t) = (r(t)X(t) - C(t)) dt + \pi^T(t) \beta(t) dt + \pi^T(t) \sigma(t) dW(t) + \pi^T(t) \rho(t) dQ(t), \]
where \( \pi(t) \) is a vector whose every component is 1. And this is the Merton’s model.

**Definition 1.1.** A **portfolio** process \( \pi(t) = (\pi_1(t), \ldots, \pi_{d+1}(t))^T \) is an \( \mathcal{F}_t \)-predictable, \( \mathbb{R}^{d+1} \)-valued process on \( (\Omega, \mathcal{F}, \mathbb{P}) \) for which
\[
\sum_{i=1}^{d+1} \int_0^t \pi_i^2(t) dt = \int_0^t \| \pi(t) \|^2 dt < \infty, \text{ a.s.} \tag{1.9}
\]
It represents the investment that the investor maintains in the \( d + 1 \) stocks. A **consumption process** \( C(t), 0 \leq t \leq T \) is non-negative predictable w.r.t. \( \{\mathcal{F}_t\} \), and satisfies
\[
\int_0^T C(t) dt < \infty, \text{ a.s.} \tag{1.10}
\]

**Remark.** We allow \( \pi_i(t) \) to take negative values corresponding to short selling the \( i \)th stock.

In addition, if we suppose that each stock pays out a continuous stream of dividends determined by a **dividend rate process** \( \delta(t), 0 \leq t \leq T, i = 1, \ldots, d + 1 \), i.e., the dividend paid out for each unit of money invested in the stock, then the investor’s wealth process \( X \) satisfies
\[
\begin{align*}
\frac{dX(t)}{dt} &= (r(t)X(t) - C(t)) dt + \pi^T(t) \beta(t) dt + \pi^T(t) \sigma(t) dW(t) + \pi^T(t) \rho(t) dQ(t), \\
X(0) &= x
\end{align*} \tag{1.11}
\]
where the process \( \delta(t) \) is assumed to be predictable and bounded, similar to the processes \( \beta \) and \( r \). And the **instantaneous expected return** from investment in stock \( i \) is \( b_i(t) + \delta_i(t) \), the **risk premium** is \( b_i(t) + \delta_i(t) - r(t) \).

Define the \( \mathbb{R}^{d+1} \)-valued process of relative risk as
\[
\theta(t) \triangleq (\overline{\pi}(t))^{-1} [b(t) + \delta(t) - r(t)] = \left[ \begin{array}{c} \theta_W(t) \\ \theta_Q(t) \end{array} \right],
\]
where \( \theta_W(t) \) is an \( \mathbb{R}^d \)-valued process and \( \theta_Q(t) \) is an \( \mathbb{R}^1 \)-valued process. The process \( \theta(t) \) is bounded, measurable and predictable w.r.t. \( \mathcal{F}_t \), by the assumptions on \( b, \delta, r \) and \( \overline{\pi} \). It represents the relative risk-premium as implied by stock returns and stock volatilities. Define the following processes
\[
\hat{W}(t) \triangleq W(t) + \int_0^t \theta_W(s) ds, \quad \hat{Q}(t) \triangleq Q(t) + \int_0^t \theta_Q(s) ds,
\]

then (1.3) can be cast as
\[
dP_1(s) = P_1(s-) \left[ r(s) ds - \delta(s) ds + \sigma(s) d\hat{W}(s) + \rho(s) d\hat{Q}(s) \right], \quad P_1(0) = p_0.
\]  

We now introduce the boundedness condition on the relative risk process of the jumps.

**Assumption 3.** We require the following inequalities
\[
\theta_Q(t) < \lambda(t) \quad \text{and} \quad \inf_{0 \leq t \leq T} \left\{ 1 - \theta_Q(t)/\lambda(t) \right\} \geq a > 0, \quad \text{a.s.}
\]  

**Remark.** This means that the relative risk-premium process is bounded from above. This condition will be required to construct the risk-neutral measure. Intuitively, the risk-neutral measure changes the drift in the stock prices to \( r - \delta \). If the risk-premium is positive, then the drift \( b \) is greater than \( r - \delta \) and must be brought down by the new measure. In this case, if the relative risk-premium is higher than the rate at which jumps are contributing to the upward drift, we cannot get a measure to bring the stock price drift down to the level we want. Assumption 3 is also necessary in mathematics.

Consider the following equations
\[
dZ_W(t) = -Z_W(t) \theta_W^T(t) dW(t), \quad Z_W(0) = 1,
\]
\[
dZ_Q(t) = Z_Q(t-) \mu(t) dQ(t), \quad Z_Q(0) = 1,
\]

where
\[
\mu(t) = -\theta_Q(t)/\lambda(t).
\]

Note \( \mu(t) \) is well defined since \( \lambda(t) > 0 \). Furthermore, \( \mu(t) > -1 \) since (1.14).

The solutions of (1.15) and (1.16) are given as follow
\[
Z_W(t) = \exp \left\{ -\int_0^t \theta_W^T(s) dW(s) - \frac{1}{2} \int_0^t \left\| \theta_W(s) \right\|^2 ds \right\},
\]
\[
Z_Q(t) = \prod_{s=1}^{N(t)} (\mu(t_n) + 1) \exp \left\{ -\int_0^t \mu(s) \lambda(s) ds \right\}.
\]

Here we can see in mathematical sense why we should have \( 1 + \mu(t) > 0 \). Otherwise, (1.19) would be invalid.

Let us denote \( Z(t) \triangleq Z_W(t)Z_Q(t) \), and from Itô’s formula we obtain
\[
dZ(t) = -Z(t-) \left[ \theta_W^T(t) dW(t) - \mu(t) dQ(t) \right], \quad Z(0) = 1,
\]

In other words, \( Z(t) \) satisfies equation (1.20).

The next lemma describes the risk-neutral measure.
Lemma 1.1[3, 4]. The process $Z$ is a $P$-martingale with $E[Z(T)] = 1$. Define an auxiliary probability measure on $(\Omega, \mathcal{F}_T)$ as $\tilde{P}(A) \equiv E[Z(T)1_A]$, $A \in \mathcal{F}_T$. Then $\tilde{W}$ and $\tilde{Q}$ are martingales under $\tilde{P}$. In particular, the jump process $N$ admits $(\tilde{P}, \mathcal{F}_T)$-stochastic intensity $\tilde{\lambda}(t) = (\mu(t) + 1)\lambda(t)$.

We can obtain the solution of (1.13) in explicit form

$$
\beta(t)P_i(t) = p_i \exp\left\{-\int_0^t \delta_i(s)ds\right\} \prod_{n=1}^{N(t)} (1 + \rho_i(t_n)) \cdot \exp\left\{-\int_0^t \rho_i(s)\tilde{\lambda}(s)ds\right\} 
\cdot \exp\left\{\int_0^t \sigma_i(s)d\tilde{W}(s) - \frac{1}{2}\int_0^t \|\sigma_i(s)\|^2 ds\right\}. 
$$

(1.21)

Also, the solution of (1.3) can be expressed as

$$
\exp\left\{-\int_0^t b_i(s)ds\right\} P_i(t) = p_i \prod_{n=1}^{N(t)} (1 + \rho_i(t_n)) \cdot \exp\left\{-\int_0^t \rho_i(s)\tilde{\lambda}(s)ds\right\} 
\cdot \exp\left\{\int_0^t \sigma_i(s)dW(s) - \frac{1}{2}\int_0^t \|\sigma_i(s)\|^2 ds\right\}. 
$$

(1.22)

From (1.21), we can see that the expected appreciation rate is exactly $r - \delta_i$ for the $i$th stock, and the total expected return from investment in any stock is equal to the interest rate $r$, under the risk-neutral measure $\tilde{P}$. But under $P$, the instantaneous appreciation rate of $i$th stock is $b_i$, and the instantaneous expected return from investment in stock $i$ is $b_i(t) + \delta_i(t)$. In practice, $b_i$ is very difficult to estimate, whereas, $b_i$ can be observed directly. This shows that the risk-neutral measure is not only helpful in theoretical research, but also useful in practical handling.

Under $\tilde{P}$, in terms of $\tilde{W}$ and $\tilde{Q}$, (1.11) can be rewritten as

$$
dX(t) = (r(t)X(t) - C(t))dt + \pi^T(t)\sigma(t)d\tilde{W}(t) + \pi^T(t)\rho(t)d\tilde{Q}(t), \quad X(0) = x. 
$$

(1.23)

A solution to this differential equation with $X(0) = x \geq 0$ is

$$
\beta(t)X(t) + \int_0^t \beta(s)C(s)ds = x + \int_0^t \beta(s)\pi^T(s)\sigma(s)d\tilde{W}(s) + \int_0^t \beta(s)\pi^T(s)\rho(s)d\tilde{Q}(s),
$$

so

$$
\tilde{M}(t) \equiv \beta(t)X(t) + \int_0^t \beta(s)C(s)ds 
$$

(1.24)

is a local martingale under $\tilde{P}$, by Lemma 1.1. This is because $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s., and the process $\beta$ and $\pi$ are bounded uniformly.

Definition 1.2. A portfolio and consumption processes pair $(\pi, C)$ is admissible w.r.t. initial wealth $x \geq 0$ if the corresponding wealth process $X$ satisfies
\[ X(t) \geq 0, 0 \leq t \leq T, \text{ a.s., and equation (1.11) or (1.23) or (1.24)}. \]

Denote the class of admissible pair w.r.t. initial wealth \( x \) as \( \mathcal{A}(x) \).

For any \((\pi, C) \in \mathcal{A}(x)\), the nonnegative local \( \tilde{P} \)-martingale \( \tilde{M} \) of (1.25) is bounded from below and is hence a nonnegative \( \tilde{P} \)-supermartingale (cf. [4, 21-25, 56]). According to the Lemma below, \( \beta(t)X(t) \) is also a supermartingale.

**Lemma 1.3.** Assume that \( Y(t) \) is a supermartingale (or martingale), and \( A(t) \) is a monotonic increasing process, then \( X(t) - A(t) \) is a supermartingale.

**Proof.**

\[
E[Y(t) - A(t) | \mathcal{F}_s] \leq \text{ or } = \ Y(s) - E[A(t) - A(s) | \mathcal{F}_s] = Y(s) - E[A(t) - A(s) + A(s) | \mathcal{F}_s] \\
= Y(s) - A(s) - E[A(t) - A(s) | \mathcal{F}_s] \leq Y(s) - A(s), \text{ for any } 0 \leq s \leq t \leq T.
\]

Let \( \tau_0 = T \wedge \inf \{ t \in [0, T] : X(t) = 0 \} \), then \( X(\tau) = 0, \tau_0 \leq \tau \leq T \) holds on \( \{ \tau_0 < T \} \) a.s. Indeed, \( \beta(t)X(t) = \tilde{M}(t) - \int_0^t \beta(s)C(s)ds \) is a supermartingale by Lemma 1.3, then for any \( \tau \geq \tau_0 \), we have that

\[
0 \leq E[\beta(\tau)X(\tau)I_{[\tau_0, T]}] \leq E[\beta(\tau_0)X(\tau_0)I_{[\tau_0, T]}] = 0.
\]

And it comes out that \( E[\beta(\tau)X(\tau)I_{[\tau_0, T]}] = 0 \). For \( \beta(\tau)X(\tau) \geq 0 \) a.s., then we get \( X(\tau) = 0 \), a.s. on \( \{ \tau_0 < T \} \). If \( \tau_0 < T \), we interpret that bankruptcy occurred.

Applying Itô’s rule to the product of the processes \( Z(t) \) and \( \beta(t)X(t) \) of (1.20) and (1.24) gives us

\[
d\left[ \beta(t)X(t) \right] = \beta(t)X(t)dZ(t) + \beta(t)\pi^T(t)dW(t) + \theta(t)X(t)dQ(t) + \beta(t)\pi^T(t)\theta(t)dt.
\]

Denote \( \zeta(t) \triangleq \beta(t)Z(t) \). We can assume that paths of \( \beta(t)X(t) \) are RCLL (cf. P33, [19]). Then from (1.26) and the properties of stochastic integral, we have that

\[
M(t) \triangleq \zeta(t)X(t) + \int_0^t \zeta(s)C(s)ds
\]

is bounded a.s. on \([0, T]\), thus \( \zeta(t)X(t) \) is also bounded a.s. on \([0, T]\). Consequently, \( M(t) \) is a nonnegative local \( \tilde{P} \)-martingale, and is a nonnegative \( \tilde{P} \)-supermartingale.
ζ acts as a \textit{deflator}, it has the meaning of that multiplication by ζ(t) converts wealth held at time t to the equivalent amount of wealth held at time zero. Applying (1.5), (1.20) and Itô’s formula leads to the linear stochastic differential equation

\begin{equation}
\begin{aligned}
d\zeta(t) &= d\beta(t)Z(t) = -\zeta(t-)[r(t)dt + \theta W(t) - \mu(t)dQ(t)] .
\end{aligned}
\end{equation}

Lemma 1.4. Suppose that \( g(t) \) is a measurable, \( \mathcal{F}_t \)-adapted process, and \( \mathbb{E}\int_0^T |g(s)|ds < \infty \), then

\[ \mathbb{E}\int_0^T g(s)ds = E\int_0^T Z(s)g(s)ds \quad \text{and} \quad \mathbb{E}\int_0^T |Z(s)g(s)|ds < \infty, \quad 0 \leq t \leq T . \]

Proof. According to Fubini theorem,

\[ \infty > \mathbb{E}\int_0^T |g(s)|ds = \int_0^T E|g(s)|ds = \int_0^T E[Z(s)g(s)]ds = E\int_0^T |Z(s)g(s)|ds . \]

Using Fubini again gives

\[ \mathbb{E}\int_0^T g(s)ds = \int_0^T \mathbb{E}g(s)ds = \int_0^T E[Z(s)g(s)]ds = E\int_0^T |Z(s)g(s)|ds , \quad \text{where} \quad t \in [0,T] . \]

In the proof of Lemma 1.4, we have used the following fact

Lemma 1.5. If \( Y \) is an integrable, measurable and \( \mathcal{F}_t \)-adapted process, then we have \[ \mathbb{E}_i[Y(t) : A] = \mathbb{E}_T[Y(t) : A] \quad \text{for any} \quad t \in [0,T] \quad \text{and any} \quad A \in \mathcal{F}_i . \]

Here, we denote \[ \mathbb{E}_i(\xi : A) \triangleq E[Z(s)\xi : A] \quad \text{for any integrable random variable} \quad \xi , \quad \text{any} \quad s \in [0,T] \quad \text{and any} \quad A \in \mathcal{F}_i . \]

Proof. For any \( t \in [0,T] , \)

\[ \mathbb{E}_i[Y(t) : A] = E[Z(T)Y(t) : A] = E[E[Z(T)Y(t) \mid \mathcal{F}_i] : A] = E[Z(t)E[Y(t) \mid \mathcal{F}_i] : A] \]

\[ = E[Z(t)Y(t) : A] = \mathbb{E}_i[Y(t) : A] , \quad \text{where} \quad A \in \mathcal{F}_i \quad \text{is arbitrary} . \]

Lemma 1.5 shows the \textit{consistency} of \( \tilde{P} \) (or \( \tilde{\mathbb{E}} \)).

Lemma 1.6. Under the conditions of Lemma 1.4,

\[ E\left(\int_0^T Z(s)g(s)ds \mid \mathcal{F}_i\right) = E\left(Z(T)\int_0^T g(s)ds \mid \mathcal{F}_i\right) = Z(t)\mathbb{E}\left(\int_0^T g(s)ds \mid \mathcal{F}_i\right) , \quad 0 \leq t \leq T . \]

Proof. For any \( A \in \mathcal{F}_i \),

\[ E\left[Z(T)\int_0^T g(s)ds : A\right] = E\left[\int_0^T g(s)ds : A\right] = \int_0^T E[g(s) : A]ds = \int_0^T E[Z(s)g(s) : A]ds \]

\[ = E\left[\int_0^T Z(s)g(s)ds : A\right] , \]

thus we have shown the first equality of this lemma. The second equality can be derived from Bayes formula.

Remark. Bayes formula is that: for \( 0 \leq s < t \leq T \) and any \( \mathcal{F}_i \)-measurable, \( \tilde{P} \)-integrable random variable, \( Z(s)\tilde{E}[Y \mid \mathcal{F}_i] = E[YZ(t) \mid \mathcal{F}_i] \). Indeed, for any \( A \in \mathcal{F}_i \), the Bayes formula can follow from
From Lemma 1.4 and the supermartingale property of $M$ and $\tilde{M}$, it follows that
\[
E \left[ \zeta(t)X(t) + \int_0^t \zeta(s)C(s)ds \right] = \tilde{E} \left[ \beta(t)X(t) + \int_0^t \beta(s)C(s)ds \right] \leq x. \tag{1.29}
\]

Then we get necessary condition for $(\pi, C) \in A(x)$ (admissibility)
\[
E \int_0^t \zeta(s)C(s)ds = \tilde{E} \int_0^t \beta(s)C(s)ds \leq x. \tag{1.30}
\]
\[
E [\zeta(t)X(t)] = \tilde{E} (\beta(t)X(t)) \leq x. \tag{1.31}
\]

With the interpretation of the process $\zeta$ as a deflator, the inequality (1.29) has the significance of a budget constraint; it mandates that “the expected total value of current wealth and consumption-to-date, both deflated down to $t = 0$, does not exceed the initial capital”.

**Definition 1.7.** For every given $x \geq 0$, let
(i) $\mathcal{C}(x)$ (respectively, $\mathcal{D}(x)$) denote the class of consumption processes $C$ which satisfy (1.30) (respectively, (1.30) as an equality);
(ii) $\mathcal{L}(x)$ (respectively, $\mathcal{M}(x)$) denote the class of nonnegative random variables $B$ on $(\Omega, \mathcal{F}, \tilde{P})$ which satisfy
\[
E (B \zeta(T)) = \tilde{E} (B \beta(T)) \leq x \tag{1.32}
\]
(respectively, (1.32) as an equality);
(iii) $\mathcal{P}(x)$ denote the class of portfolio processes $\pi$ such that $(\pi, 0) \in A(x)$ and the corresponding terminal wealth $X(T) \in M(x)$.

From the inequality (1.29), we deduce
\[
(\pi, C) \in A(x) \Rightarrow C \in \mathcal{C}(x) \text{ and } X(T) \in \mathcal{L}(x).
\]

We should show that $\mathcal{C}(x)$ consists of exactly those “reasonable” consumption processes, for which an investor, starting out with wealth $x$ at time $t = 0$, is able to construct a portfolio that will avoid debt (i.e., negative wealth) on $[0, T]$, almost surely. From (1.29), we have shown that the conditions (1.30) and (1.31) are necessary for admissibility, next we will see they are also sufficient in a sense for admissibility.

**Theorem 1.8.** (Portfolio determined by initial endowment and consumption)
1. For any given $C \in \mathcal{C}(x)$, there exists a portfolio process $\pi$ such that $(\pi, C) \in A(x)$;
2. Let $\xi$ be a nonnegative, $\mathcal{F}_T$-measurable random variable, $C_1$ is a consumption process such that
E\left[ \zeta(t)\xi + \int_0^t \zeta(s)C_1(s)ds \right] = \hat{E}\left( \beta(t)\xi + \int_0^t \beta(s)C_1(s)ds \right) \leq x, \text{ then there exists a portfolio } \pi \text{ such that } (\pi, C_1) \in \mathcal{A}(x) \text{ with the terminal wealth } X_1(T) \geq E(\zeta(T)\xi) / \zeta(T) \left( \beta(T)x(T) \geq E(\beta(T)\xi) \right).

**Proof. 1. Proof I.** Let \( D \triangleq \int_0^T C(s)\zeta(s)ds \), define the nonnegative process as
\[
\Omega(t) \triangleq E(D|\mathcal{F}_T) - E[D] + x - \int_0^t C(s)\zeta(s)ds, \tag{1.33}
\]
and assume that \( E(D|\mathcal{F}_T) \) is RCLL, then the \((P, \mathcal{F}_t)\)-martingale \( u(t) \triangleq E(D|\mathcal{F}_T) - E[D], 0 \leq t \leq T \) can be represented as a stochastic integral w.r.t. \((W, Q)\), i.e., there exist \{\mathcal{F}_t\}\-predictable \( \mathbb{R}^d \)-valued process \( \eta_W \) and \{\mathcal{F}_t\}\-predictable \( \mathbb{R}^1 \)-valued process \( \eta_Q \), with
\[
\int_0^T \left( \|\eta_W(s)\|^2 + \|\eta_Q(s)\|^2 \right)ds < \infty, \text{ a.s., such that }
\]
\[
u(t) = \int_0^t \eta_W(s)dW(s) + \int_0^t \eta_Q(s)dQ(s). \tag{1.34}
\]
This is because that the family of martingales \{W_j : 1 \leq j \leq d\} and \( Q \) has the predictable representation property on the product space \([4, 7, 44]\). Let
\[
\zeta(t) - \pi(t)\sigma(t) - X(t)\beta(t) + \chi(t) = \eta_W(t), \quad \zeta(t) - \pi(t)\sigma(t) - X(t)\beta(t) + \chi(t) + \chi(t) = \eta_Q(t),
\]
then we get from the above two equations
\[
\pi(t) = \left[ \pi(t)^{\top} \right]^{-1} \left[ \begin{bmatrix} \eta_W(t) + \zeta(t)X(t)\beta(t) \end{bmatrix} \right], \tag{1.35}
\]
so that (1.33) becomes (1.26) when we make the identification \( \Omega(t) \triangleq \zeta(t)X(t) \). The process defined in (1.35) is \{\mathcal{F}_t\}\-predictable and satisfies \( \int_0^T \|\pi(t)\|^2 dt < \infty, \) a.s. The corresponding wealth process \( X \) is given by
\[
\zeta(t)X(t) = E(\int_0^T C(s)\zeta(s)ds|\mathcal{F}_T) - E[D] + x. \text{ Furthermore, }
\]
\[
X(T) = (x - ED) / \zeta(T) \geq 0. \tag{1.36}
\]

**Proof II.** Let \( D \triangleq \int_0^T C(s)\beta(s)ds \), and define a nonnegative process by
\[
\Omega(t) \triangleq \hat{E}[D|\mathcal{F}_T] - \hat{E}D + x - \int_0^t C(s)\beta(s)ds. \tag{1.33'}
\]
From Bayes Formula, we obtain \( m(t) \triangleq \hat{E}(D|\mathcal{F}_T) - \hat{E}D = \frac{E[DZ(T)|\mathcal{F}_T]}{Z(t)} - E(DZ(T)) \).
Assume that paths of martingale \( N(t) \triangleq E(DZ(T)|\mathcal{F}_t) \) are RCLL, a.s., then there exist \( \{\mathcal{F}_t\} \)-predictable \( \mathbb{R}^d \)-valued process \( Y_W \) and \( \{\mathcal{F}_t\} \)-predictable \( \mathbb{R}^1 \)-valued process \( Y_Q \), and
\[
\int_0^T (\|Y_W(s)\|^2 + \|Y_Q(s)\|^2) \, ds < \infty, \text{ a.s., such that}
\]
\[
N(t) = E(DZ(T)) + \int_0^T Y_W^T(s) \, dW(s) + \int_0^T Y_Q(s) \, dQ(s).
\]

Applying Itô’s formula on \( N(t)/Z(t) \), with (1.20) taking into account yields
\[
dm(t) = d\left( N(t)/Z(t) \right) = \frac{1}{Z(t)} Y_W^T(t) \, dW(t) - N(t)(Z(t))^{-2} \left[ -Z(t)\theta_W^T(t) \, dW(t) \right]
+ \frac{1}{2} 2N(t)(Z(t))^{-3} \left[ Z(t)\theta_W^T(t) \right] dt - (Z(t))^{-2} \left[ -Z(t)\theta_W^T(t)Y_W(t) \right] dt
+ \left[ \frac{N(t) + Y_Q(t)}{Z(t)(1 + \mu(t))} - \frac{N(t)(Z(t))^{-1}}{Z(t)} \right] \, dQ(t) + \left[ \frac{N(t) + Y_Q(t)}{Z(t)} - \frac{N(t)}{Z(t)} \right] \, dQ(t) - \frac{N(t)}{Z(t)} \cdot Z(t)\mu(t) \right] \lambda(t) dt
= \frac{1}{Z(t)} (Y_W(t) + N(t)\beta_W(t))^T \, dW(t) + \frac{1}{Z(t)} (Y_W(t) + N(t)\beta_W(t))^T \theta_W(t) \, dW(t)
+ \frac{Y_Q(t) - N(t)\mu(t)}{Z(t)(1 + \mu(t))} \, dQ(t) - \frac{\mu(t)(Y_Q(t) - N(t)\mu(t))}{Z(t)(1 + \mu(t))} \right) \, \lambda(t) dt
= \frac{1}{Z(t)} (Y_W(t) + N(t)\beta_W(t))^T \, dW(t) + \frac{Y_Q(t) - N(t)\mu(t)}{Z(t)(1 + \mu(t))} \, dQ(t). \quad (1.34')
\]

Let \( \beta(t)\pi^T(t)\sigma(t) = \frac{1}{Z(t)} (Y_W(t) + N(t)\beta_W(t))^T \), \( \beta(t)\pi^T(t)\rho(t) = \frac{Y_Q(t) - N(t)\mu(t)}{Z(t)(1 + \mu(t))} \).

Then obtain that
\[
\pi(t) = (\pi^T(t))^{-1} \left[ \frac{(Y_W(t) + N(t)\beta_W(t))/\xi(t)}{(Y_Q(t) - N(t)\mu(t))/\zeta(t)(1 + \mu(t))} \right]. \quad (1.35')
\]

When we set \( \Omega(t) \triangleq \beta(t)X(t) \), (1.33') will become to (1.24). \( \pi(t) \) defined in (1.35') is \( \{\mathcal{F}_t\} \)-predictable and satisfies \( \int_0^T \|\pi(t)\|^2 < \infty \), a.s. And the corresponding wealth process is decided by
\[
\beta(t)X(t) = \hat{E}(\int_0^T C(s)\beta(s) \, ds |\mathcal{F}_t) - \hat{E}[D] + x.
\]

2. Clearly, \( C_1(t) \in \mathcal{B}(x) \) by the assumption. And from the previous proof of 1, there exists a portfolio process \( \pi_1 \) such that \( (\pi_1, C_1) \in \mathcal{A}(x) \), and \( X_1(T) \zeta(T) \overset{(1.36)}{=} x - \hat{E} \int_0^T \zeta(s)C_1(s) \, ds \geq E(\xi \cdot \zeta(T)) \).

And for similar reasons, we have
\[
\beta(T)X_1(T) \geq \hat{E}(\xi \cdot \beta(T)).
\]
Remark. In the previous theorem, whether introducing the “deflator” or “discount factor” in the proofs, we can always obtain the same conclusions. But in the Proof II, we should represent the martingale as the sum of two stochastic integrals with respect to \( \hat{W} \) and \( \hat{Q} \) [3, 4]. This representation cannot be obtained from a direct application of martingale representation theorem to the \( \hat{P} \)-martingale, since the filtration \( \{ F_t \} \) is the augmentation (under \( P \) or \( \hat{P} \)) of \( \{ F_t^W \otimes F_t^N \} \), not of \( \{ F_t^{\hat{W}} \otimes F_t^{\hat{N}} \} \) (cf. P375, [25]).

We say that two measurable stochastic processes \( A, B \) on \([0, T]\) are equivalent, if \( A(t, \omega) = B(t, \omega) \) holds for \( \lambda \otimes P \)-a.e. \((t, \omega) \in [0, T] \times \Omega \). Here \( \lambda \) is Lebesgue measure on \([0, T]\).

Proposition 1.9. For each \( C \in \mathcal{D}(x) \) defined in Definition 1.7, we have the following:

(i) The corresponding wealth process \( X \) satisfies \( X(T) = 0 \), a.s.;

(ii) The process \( M \) of (1.27) (or respectively, \( \hat{M} \) of (1.25)) is a \( P \) (or \( \hat{P} \))-martingale. Furthermore,

\[
\zeta(t)X(t) = E \left( \int_0^T \zeta(s)C(s)ds \mid F_t \right) \quad \text{(or, } \beta(t)X(t) = \hat{E} \left( \int_0^T \beta(s)C(s)ds \mid F_t \right) , 0 \leq t \leq T ;
\]

(iii) The portfolio \( \pi \) is unique up to equivalence, and the wealth process \( X \) is also unique.

Proof. For each given \( C \in \mathcal{D}(x) \subset \mathcal{C}(x) \), there exists a portfolio process \( \pi \) such that \( (\pi, C) \in \mathcal{A}(x) \) by Theorem 1.8. We have from (1.29) the inequality

\[
E(\zeta(T)X(T)) \leq x - E\int_0^T \zeta(s)C(s)ds = 0 , \text{ which proves (i), as well as}
\]

\[
EM(T) = E \int_0^T \zeta(s)C(s)ds = x = EM(0) . \text{ And } EM(t) \leq EM(t) \leq EM(0), 0 \leq t \leq T , \text{ since } M \text{ is a supermartingale, thus } EM(t) \equiv x, \text{ and this leads to } M(t) \text{ being a } P \text{-martingale. Similarly, } \hat{M}(t)
\]

is also a \( \hat{P} \)-martingale.

As for uniqueness, let \( \pi_1, \pi_2 \) be two arbitrary portfolio processes such that both \( (\pi_1, C) \) and \( (\pi_2, C) \) are in \( \mathcal{A}(x) \), and let \( X_1, X_2 \) be the corresponding wealth processes and \( \hat{M}_1, \hat{M}_2 \) the corresponding martingales from (1.25). Then we obtain by (1.24) that

\[
(\hat{M}_1 - \hat{M}_2)(t) = \beta(t)(X_1(t) - X_2(t)) = \int_0^t \beta(s)(\pi_1(s) - \pi_2(s))^T \sigma(s)d\hat{W}(s) + \int_0^t \beta(s)(\pi_1(s) - \pi_2(s))^T d\hat{Q}(s).
\]

Moreover, \( \hat{M}_1(T) = \hat{M}_2(T) = \int_0^T \beta(s)C(s)ds \) and \( \hat{M}_1 - \hat{M}_2 \) is a \( \hat{P} \)-martingale, then

\[
(\hat{M}_1 - \hat{M}_2)(t) = \hat{E} \left( (\hat{M}_1 - \hat{M}_2)(T) \mid F_t \right) = \hat{E}(0 \mid F_t) \equiv 0 , \text{ and then}
\]


\[
0 \equiv \left( M_1 - M_2 \right)(t) = \int_0^t \beta^2(s) \left( \pi_1(s) - \pi_2(s) \right)^T \sigma(s) \left( \pi_1(s) - \pi_2(s) \right)^T \rho(s) d\lambda(s) ds.
\]

Thus, \((\pi_1(t) - \pi_2(t))^T \sigma(t) = 0\), so \(\pi_1(t) - \pi_2(t) = 0\), a.s., a.e., since \(\sigma\) is nondegenerate. Then it follows that \(\pi_1, \pi_2\) are equivalent.

Furthermore, it follows from the previous that \(0 \equiv (\bar{M}_1 - \bar{M})(t) = \beta(t)(X_1(t) - X_2(t))\), i.e., the wealth process \(X\) and martingale \(\bar{M}\) corresponding to \(C \in \mathcal{D}(x)\) are all unique.

In Theorem 1.8, we show that one of the wealth process corresponding to \(C \in \mathcal{C}(x)\) can be expressed as \(\zeta(t)X(t) = E\left( \int_t^T \zeta(s)C(s)ds | \mathcal{F}_t \right) - E\left( \int_0^T \zeta(s)C(s)ds \right) + x\), and now we have shown that the wealth process corresponding to \(C \in \mathcal{D}(x)\) is unique, thus it certainly can be expressed as
\[
\zeta(t)X(t) = E\left( \int_t^T \zeta(s)C(s)ds | \mathcal{F}_t \right), \quad 0 \leq t \leq T.
\]

**Remark.** Theorem 1.8 shows that: for any \(C \in \mathcal{C}(x)\), we can construct a portfolio \(\pi\), such that the corresponding \(M\) (or \(\bar{M}\)) is \(P\) (or \(\bar{P}\))-martingale. But this is valid only under some special construction; Now Proposition 1.9 shows that: for any \(C \in \mathcal{D}(x)\), \(X, \pi, M\) (or \(\bar{M}\)) can be uniquely determined, and the \(M\) (or \(\bar{M}\)) is martingale, not depending on the constructions.

**Theorem 1.10.** (Portfolio determined by initial endowment and terminal wealth)

For each \(B \in \mathcal{L}(x)\), there exists a pair \((\pi, C) \in \mathcal{A}(x)\) such that the corresponding wealth process \(X\) satisfies \(X(T) = B\), a.s.

**Proof.**

**Proof I.** Let \(Q \triangleq B \zeta(T)\), and define that
\[
\zeta(t)X(t) \triangleq E\left( Q | \mathcal{F}_t \right) + (x - EQ) \left( 1 - t/T \right) \triangleq x + m(t) - pt,
\]
here \(m(t) = E(Q | \mathcal{F}_t) - EQ\), \(p = (x - EQ)/T\). By the similar argument in the Proof I of Theorem 1.8.1, we obtain a stochastic integral representation of \(P\)-martingale \(m(t)\) in the form (1.34), and then (1.37) can be cast in the form (1.26) if we take \(\pi\) as in (1.35) and \(C(t) = p/\zeta(t)\).

**Proof II.** Let \(Q \triangleq B \beta(T)\), and define
\[
\beta(t)X(t) \triangleq \hat{E}\left( Q | \mathcal{F}_t \right) + (x - \hat{E}Q) \left( 1 - t/T \right) \triangleq x + m(t) - pt,
\]

where \( m(t) = \tilde{E}(Q|\mathcal{F}_t) - \tilde{E}Q, \rho = (x - \tilde{E}Q)/\tilde{T}, \ X(0) = x, \ X(T) = B, \text{ a.s.} \) By analogy with Proof II of Theorem 1.8.1, a stochastic integral representation of \( \tilde{F} \)-martingale \( m(t) \) is obtained in the form (1.34'), and (1.37') can be cast in the form (1.24) if we take \( \pi \) as in (1.35') and \( C(t) = \rho f(\beta(t)). \)

**Proposition 1.11.** For any \( B \in \mathcal{M}(x) \), we have the following

(i) The pair \((\pi, C) \in \mathcal{A}(x)\) is uniquely determined up to equivalence, and \( C(t) \equiv 0, \text{ a.s., a.e.} \) And the corresponding wealth process \( X \) is also unique with \( X(T) = B \). Here \( \pi \in \mathcal{P}(x) \);

(ii) The process \( M(t) \) (or \( \tilde{M}(t) \)) is \( P \) (or \( \tilde{P} \))-martingale. In particular, the corresponding wealth process \( X \) is given by \( \zeta(t)X(t) = E (B \zeta(T)|\mathcal{F}_t) \) (or \( \beta(t)X(t) = \tilde{E} (B \beta(T)|\mathcal{F}_t) \)), \( 0 \leq t \leq T \).

Proof. From (1.29), we have \( E \int_0^T \zeta(s)C(s)ds \leq x - E (B \zeta(T)) = 0 \), which verifies \( C(t) \equiv 0, \text{ a.s., a.e.} \), and uniqueness of \( C(s) \), as well as \( EM(T) = E (B \zeta(T)) = x = EM(0) \), which shows that \( M \) is a martingale since supermartingale \( M \) has a constant expectation value. Other conclusions such as uniqueness of \( \pi \) and \( X \), etc can be shown in similar ways as Proposition 1.9.

**Remark.** Theorem 1.10 and the relation (1.31) show that \( \mathcal{L}(x) \) consists of precisely those “levels of terminal wealth” which are attainable from the initial endowment \( x \geq 0 \), by using some portfolio-consumption pair that avoids debt. In other words, \( \mathcal{L}(x) \) is an attainable set of \( X \) with \( X(0) = x \). The terminology here is due to Pliska [43]. Proposition 1.11 shows that the “extreme” elements of \( \mathcal{L}(x) \) are attainable by strategies that mandate zero consumption.

**Definition 1.12.** Define an **arbitrage opportunity** as a portfolio \( \pi \) such that

(i) \((\pi, 0) \in \mathcal{A}(0)\), and (ii) The wealth process \( X \) corresponding to \((\pi, 0)\) and the initial capital \( X(0) = x = 0 \), satisfies \( P (X(T) > 0) > 0 \).

In other words, an arbitrage opportunity is an admissible investment strategy with zero initial capital and zero consumption, whose terminal wealth is positive with positive probability. It is also called a “free lunch” sometimes.

Our model excludes arbitrage opportunities. Indeed, the necessary conditions (1.30) and (1.31) with initial capital \( x = 0 \) yield \( C(t) \equiv 0 \) a.e., a.s. and \( X(T) = 0 \) a.s.

**2. Valuation of European contingent claims**

A **European contingent claim** (Abbriviated, **ECC**) is a financial instrument consisting of a
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dividend payoff rate $g(t), t \in [0,T]$, and a terminal liquidation payoff value $f$. Here, $g$ is
nonnegative, bounded, measurable and $\mathcal{F}_t$-adapted process, and $f$ is a nonnegative, $\mathcal{F}_T$-
measurable random variable. In addition, $g$ and $f$ are assumed to satisfy
\[
E\left(\beta(T)f + \int_0^T \beta(s)g(s)ds\right) = E\left(\zeta(T)f + \int_0^T \zeta(s)g(s)ds\right) < \infty.
\] (2.1)

Let $x \geq 0$ be given, a pair $(\pi,C) \in \mathcal{A}(x)$ is called a hedging strategy against the contingent claim
$(g,f)$, if $C(t) = g(t), 0 \leq t \leq T$; and $X(T) = f$ hold a.s., where $X$ is the wealth process
associated with the pair $(\pi,C)$ and with the initial condition $X(0) = x$. Denote
\[
\mathcal{H}(x) = \{(\pi,C) : (\pi,C) \text{ a hedging strategy against the ECC } (g,f) \text{ with initial wealth } X(0) = x\}.
\]

In words, a hedging strategy $(\pi,C) \in \mathcal{H}(x)$ starts out with initial wealth $x$ and duplicates the
payoff from the ECC.

Suppose that at time $t = 0$, we sign a contract that gives us the option to buy, at the
specified time $T$ (i.e., maturity or expiration date), one share of the stock $i = 1$ at a specified
price $q$ (the contractual "exercise price"). At time $T$, if the price $P_i(T)$ of the share is below the
exercise price, then the contract is worthless to us. On the other hand, if $P_i(T) > q$, we can
exercise our option at this time by buying one share of the stock at the pre-assigned price $q$,
and then selling the share immediately in the market for $P_i(T)$, earning a profit of $P_i(T) - q$.
Thus, this contract is equivalent to a payment of $(P_i(T) - q)^+$ at maturity, it is called a European
call option. A European call option is a special case of an ECC with $g \equiv 0$ and $f = (P_i(T) - q)^+$.

How do we decide the fair price to pay at $t = 0$ for the ECC? If there exists a hedging
strategy $(\pi,C) \in \mathcal{H}(x)$ which is admissible for some $x > 0$, then an agent who not only can buy
the ECC $(g,f)$ at time $t = 0$ but also can instead invest the wealth $x$ in the financial market
according to the portfolio $\pi$ and consume at the rate $C$. At maturity, the agent can still duplicate
the payoff from buying the ECC. Consequently, the price he should be prepared to pay at $t = 0$
for the ECC cannot possibly be greater than this amount $x$. Then it is natural to define the fair
price of as the smallest value of the initial wealth, which allows constructing a hedging strategy.

Let us define the number $v \triangleq \inf\{x > 0 : \exists(\pi,C) \in \mathcal{H}(x)\}$ as the fair price at $t = 0$ for the
ECC.

Denote $Q \triangleq \zeta(T)f + \int_0^T \zeta(s)g(s)ds$, $\bar{Q} \triangleq \beta(T)f + \int_0^T \beta(s)g(s)ds$. 
Theorem 2.1.1. The fair price of the ECC \((f,g)\) is given by \(\nu \triangleq EQ = \tilde{E}Q\). There exists a unique (up to equivalence) pair \((\pi, C) \in \mathcal{H}(x)\), corresponding to a wealth process \(X\) which is also unique and given by

\[
\zeta(t)X(t) = E\left(\zeta(T)f + \int_t^T \zeta(s)g(s)ds \mid \mathcal{F}_t\right) \quad \text{or} \quad \beta(t)X(t) = \tilde{E}\left(\beta(T)f + \int_t^T \beta(s)g(s)ds \mid \mathcal{F}_t\right), \quad t \in [0,T].
\]

Furthermore, the process \(M\) in (1.27) (or \(\tilde{M}\) in (1.25)) is a \(P\) (or \(\tilde{P}\))-martingale; If \(g \equiv 0\), then

\[
\beta(t)X(t) = \tilde{E}\left(\beta(u)X(u) \mid \mathcal{F}_t\right), \quad \text{i.e.,} \quad X(t) = \tilde{E}\left(X(u) \cdot \beta(u)/\beta(t) \mid \mathcal{F}_t\right) \quad \text{for} \quad 0 \leq t < u \leq T.
\]

2. For any nonnegative number \(x \geq v\), there exists a portfolio process \(\pi_1\) such that the pair \((\pi_1, C) \in \mathcal{A}(x_1)\), and the corresponding wealth process \(X_1\) satisfies the terminal conditions

\[
\zeta(T)X_1(T) \geq E(\zeta(T)f) \quad \text{or} \quad \beta(T)X_1(T) \geq \tilde{E}(\beta(T)f).
\]

Proof. 1. Let \(C(t) = g(t)\), and consider the \(P\)-martingale \(u(t) \triangleq E(Q\mid \mathcal{F}_t) - EQ\), \(0 \leq t \leq T\). We can assume that paths of this \(P\)-martingale are RCLL, a.s (cf. P33, [19]). By similar argument in the Proof I of Theorem 1.8.1, \(u(t)\) has a stochastic integral representation in the form (1.34), and

\[
\zeta(t)X(t) = E(Q\mid \mathcal{F}_t) - \int_0^t \zeta(s)g(s)ds
\]

(2.2)
can be cast in the form (1.26) if we take \(\pi\) as in (1.35), here \(X(0) = EQ\). Now we show \(EQ\) is the fair price of the ECC \((g,f)\). By (1.29), any other hedging strategy against the contingent claim \((g,f)\) should cost at least as much as \(EQ\), and \(EQ\) is a lower bound on the fair price.

In the above proof, if we replace the deflator \(\zeta\) by the discount factor \(\beta\), and imitate the analogous argument under \(\tilde{P}\) instead of \(P\), we will get an another proof similar with the proof II of Theorem 1.8.1.

Since \(E(M(T)) = EQ = X(0) = E(M(0))\), then the mathematical expectation of supermartingale \(M\) is a constant, and it follows that \(M\) is a martingale.

The proof of the uniqueness is similar to the corresponding part in Proposition 1.9. And it comes out of (2.2) that

\[
\zeta(t)X(t) = E\left(\zeta(T)f + \int_t^T \zeta(s)g(s)ds \mid \mathcal{F}_t\right), \quad 0 \leq t \leq T.
\]

If \(g \equiv 0\), then for any \(0 \leq t < u \leq T\),

\[
\beta(t)X(t) = \tilde{E}(\beta(T)X(T) \mid \mathcal{F}_t) = \tilde{E}\left(\tilde{E}(\beta(T)X(T) \mid \mathcal{F}_u) \mid \mathcal{F}_t\right) = \tilde{E}(\beta(u)X(u) \mid \mathcal{F}_t).
\]

We may prove 2 by analogy of Theorem 1.8.2.
3. The valuation equation for derivative securities of ECC

For the hedging ECC \((g,f)\) problem, we will consider derivative securities, i.e., ECC whose payoffs and values depend on the prices of the underlying assets \(P = (P_1, \ldots, P_{d+1})\). In these cases, the price of the claim can be expressed as a function of time \(t\) and the current stock prices vector \(P = (P_1, \ldots, P_{d+1})\). The dividend stream and the terminal payoff both depend on the values of the stock prices being traded, i.e., \(g\) is a measurable function \([0,T] \times \mathbb{R}^{d+1}_+ \to \mathbb{R}_+\), and \(f\) is a measurable function \(\mathbb{R}^{d+1}_+ \to \mathbb{R}_+\). Note that \(g\) and \(f\) depend only on the current prices on the stock markets, not the path of the price process.

The previous results indicate that there exists a replicating portfolio for the derivative security. Now we would like to relate the price (value) of the derivative security and the composition of replicating portfolio to the prices of the underlying stocks. We will use hedging arguments to derive a valuation equation that must be satisfied by the ECC’s price, and indicate some kind of the equivalent relations between the solutions of the wealth process equation and of the valuation equation.

Assume that all market coefficients are deterministic, and the value of the derivative security is a function of time and the stock prices, i.e., there exists a function \(V \in C^{1,2}([0,T] \times \mathbb{R}^{d+1}_+ \to \mathbb{R}_+)\), such that \(V(t, P(t))\), \(t \in [0,T]\) is the value (price) of the derivative security at time \(t\). We will use a triple \((V(t, P(t)), g(t, P(t)), f(P(T)))\) to express the ECC derivative security, where \(V(t, P(t))\) stands for the value of the derivative security, \(P(t)\) are the current stock prices, as well as \(g(t, P(t))\) and \(f(P(T))\) stand respectively for the dividend rate and the terminal payoff of the ECC derivative security. Let \(x \geq 0\), \((\pi, C) \in \mathcal{A}(x)\) be the hedging strategy against the above-mentioned derivative security associated with the wealth process \(X(t)\). If \(X(t) \equiv V(t, P(t))\), \(C(t) = g(t, P(t))\), a.s., \(0 \leq t \leq T\), then we say that the hedging strategy \((\pi, C)\) duplicates (or replicates) completely the above-mentioned derivative security. Here \(X(t)\) is a wealth process of the hedging strategy, satisfying

\[
dX(t) = \sigma(t)dW(t) + \pi(t)d\tilde{Q}(t)g(t, P(t))dt + \pi(t)\rho(t)d\tilde{Q}(t) - g(t, P(t))dt, X(T) = f(P(T)), 0 \leq t < T. \tag{3.1}
\]

The stock prices satisfy the equation (1.13).

For any vector \(p = (p_1, \ldots, p_{d+1}) \in \mathbb{R}^{d+1}_+\), denote \(\overline{p} \triangleq [p_1(1 + \rho_1(t)), \ldots, p_{d+1}(1 + \rho_{d+1}(t))]\) for later use.
Theorem 3.1. If \((\pi, C)\) duplicates completely the ECC derivative security 
\((V(t, P(t)), g(t, P(t)), f(P(T)))\), then the value \(V(t, P(t))\) of the derivative security must satisfy the following differential-difference equation

\[
\frac{\partial V(t, P(t))}{\partial t} + \sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t)[r(t) - \delta_i(t)] + \frac{1}{2} \sum_{i,j=1}^{d+1} \frac{\partial^2 V(t, P(t))}{\partial P_i \partial P_j} P_i(t) P_j(t) \sigma_i(t) \sigma_j(t) \partial_t \nabla(t) + \left[V(t, P(t)) - V(t, P(t)) - g(t, P(t))\right] \lambda(t) = r(t)V(t, P(t)) - g(t, P(t)),
\]

\[V(T, P(T)) = f(P(T)), \quad t \in [0, T).\]

The replicating portfolio is given by

\[
\pi(t, P(t))^T = \left[\sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t) \rho_i(t) \right] \sigma^{-1}(t). \tag{3.3}
\]

Furthermore, if

\[
\sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t) \rho_i(t) = V(t, P(t)) - V(t, P(t)), \tag{3.4}
\]

then \(\pi\) is given by

\[
\pi_i(t, P(t)) = \frac{\partial V(t, P(t))}{\partial P_i} P_i(t). \tag{3.5}
\]

**Proof.** Using Itô’s formula to \(V(t, P(t))\) leads to

\[
dV(t, P(t)) = \frac{\partial V(t, P(t))}{\partial t} dt + \sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t) \left[(r(t) - \delta_i(t)) dt + \sigma_i(t) dW(t)\right] + \frac{1}{2} \sum_{i,j=1}^{d+1} \frac{\partial^2 V(t, P(t))}{\partial P_i \partial P_j} P_i(t) P_j(t) \sigma_i(t) \sigma_j(t) dt + \left[V(t, P(t)) - V(t, P(t))\right] \lambda(t) dt.
\]

Comparing this equation with the wealth process equation (3.1) of the hedging strategy against the contingent claim \((V(t, P(t)), g(t, P(t)), f(P(T)))\), it follows that (3.2) and (3.3). And with condition (3.4) in force, we obtain (3.5) by comparing the coefficients before \(dW(t)\) and \(d\lambda(t)\) in (3.6) and (3.1).

Conversely, we can derive a hedging strategy to duplicate completely the ECC \((V(t, P(t)), g(t, P(t)), f(P(T)))\) by solving a partial differential-difference equation, we show this in the following theorem. Note that \(\sigma^{-1}(t)\) is bounded uniformly since \(\sigma(t) = \sigma(t)\sigma^T(t)\) is assumed to be strongly nondegenerate.
Theorem 3.2. Suppose that $V(t, p) \in C^{1,2}([0, T] \times \mathbb{R}_+^{d+1} \to \mathbb{R}_+)$ and satisfies the following differential-difference equation

$$
\frac{\partial V(t, p)}{\partial t} + \sum_{i=1}^{d+1} \frac{\partial V(t, p)}{\partial p_i} p_i [r(t) - \delta_i(t)] + \frac{1}{2} \sum_{i,j=1}^{d+1} \frac{\partial^2 V(t, p)}{\partial p_i \partial p_j} p_i p_j \sigma_i(t) \sigma_j(t) \tag{3.7}
$$

$$+ \left[ V(t, \overline{p}) - V(t, p) - \sum_{i=1}^{d+1} \frac{\partial V(t, p)}{\partial p_i} p_i \rho_i(t) \right] \lambda(t) = r(t)V(t, p) - g(t, p), \quad t \in [0, T], \quad V(T, p) = f(p),
$$

then there exists a hedging strategy $(\pi, C)$ to duplicate completely the ECC

$$(V(t, P(t)), g(t, P(t)), f(P(T))),$$

and the replicating portfolio is given by

$$
\pi(t, P(t))^T = \left[ \sum_{i=1}^{d+1} \frac{\partial V(t, P(t^-))}{\partial p_i} P_i(t^-) \sigma_i(t), \quad V(t, \overline{P(t^-)}) - V(t, P(t^-)) \right] \sigma^{-1}(t).
$$

Furthermore, the replicating portfolio can be given by $\pi_i(t, P(t)) = \frac{\partial V(t, P(t^-))}{\partial p_i} \cdot P_i(t^-)$ provided that $V(t, p)$ satisfies

$$
\sum_{i=1}^{d+1} \frac{\partial V(t, p)}{\partial p_i} p_i \rho_i(t) = V(t, \overline{p}) - V(t, p).
$$

**Proof.** It gives us (3.5) when applying Itô’s formula on $V(t, P(t))$. With (3.7) take into consideration, we have

$$
dV(t, P(t)) = r(t)V(t, P(t))dt + \sum_{i=1}^{d+1} \frac{\partial V(t, P(t^-))}{\partial p_i} P_i(t^-) \sigma_i(t)d\overline{W}(t) \tag{3.8}
$$

$$+ \left[ V(t, \overline{P(t^-)}) - V(t, P(t^-)) \right] d\overline{Q}(t) - g(t, P(t))dt, \quad V(t, P(T)) = f(P(T)).
$$

Let $\pi(t, P(t))^T \triangleq \left[ \sum_{i=1}^{d+1} \frac{\partial V(t, P(t^-))}{\partial p_i} P_i(t^-) \sigma_i(t), \quad V(t, \overline{P(t^-)}) - V(t, P(t^-)) \right] \sigma^{-1}(t)$, then (3.8) becomes

$$(3.1), \quad \int_0^T \| \pi(t, P(t)) \|^2 dt < \infty, \text{ a.s. If we let } C(t) = g(t, P(t)) \text{ more, then the hedging strategy } \langle \pi, C \rangle \text{ duplicates completely the ECC derivative security } (V(t, P(t)), g(t, P(t)), f(P(T))).$$

The remains can be proved similarly with Theorem 3.1.

In combination of the above two theorems, we derive the necessary sufficient condition for the hedging strategy $(\pi, C)$ to duplicate completely the ECC derivative security

$$(V(t, P(t)), g(t, P(t)), f(P(T))).$$

Theorem 3.3. Suppose that $V(t, p) \in C^{1,2}([0, T] \times \mathbb{R}_+^{d+1} \to \mathbb{R}_+)$ . (I) The necessary sufficient condition for the hedging strategy $(\pi, C)$ to duplicate completely the derivative security
(V(t, P(t)), g(t, P(t)), f(P(T))) is that V(t, P(t)) satisfies the following differential-difference equation

\[
\frac{\partial V(t, P(t))}{\partial t} + \sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t)[r(t) - \delta_i(t)] + \frac{1}{2} \sum_{i,j=1}^{d+1} \frac{\partial^2 V(t, P(t))}{\partial P_i \partial P_j} P_i(t)P_j(t) \sigma_i(t) \sigma_j(t)
\]

\[
+ \left\{ V\left(t, P(t) \right) - V(t, P(t)) - \sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t) \rho_i(t) \right\} \lambda(t) = r(t)V(t, P(t)) - g(t, P(t)), \quad t \in [0, T),
\]

subject to the boundary condition V(T, P(T)) = f(P(T)). The replicating portfolio is given by

\[
\pi(t, P(t))^T = \left[ \sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t) \sigma_i(t), \ V\left(t, P(t) \right) - V(t, P(t)) \right] \sigma^{-1}(t); \]

(II) Furthermore, if V(t, P(t)) satisfies the following equation

\[
\sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t) \rho_i(t) = V(t, P(t)) - V(t, P(t)),
\]

then the necessary sufficient condition for the hedging strategy (π, C) to duplicate completely the derivative security (V(t, P(t)), g(t, P(t)), f(P(T))) is that V(t, P(t)) satisfies

\[
\frac{\partial V(t, P(t))}{\partial t} + \sum_{i=1}^{d+1} \frac{\partial V(t, P(t))}{\partial P_i} P_i(t)[r(t) - \delta_i(t)] + \frac{1}{2} \sum_{i,j=1}^{d+1} \frac{\partial^2 V(t, P(t))}{\partial P_i \partial P_j} P_i(t)P_j(t) \sigma_i(t) \sigma_j(t)
\]

\[
= r(t)V(t, P(t)) - g(t, P(t)), \quad t \in [0, T),
\]

subject to the boundary condition V(T, P(T)) = f(P(T)). And π is given by

\[
\pi(t, P(t)) = \frac{\partial V(t, P(t))}{\partial P_i} P_i(t). \]

Remark.
1. Theorem 2.1 ensures that there exists a hedging strategy to duplicate the ECC, whereas theorems here tell us when there exists a hedging strategy to duplicate completely the ECC derivative security. And the uniqueness leads to that these two hedging strategies are the same for a given ECC derivative security.

2. Although the “jumps” are “pure” non-systematic risk, this doesn’t mean that the jump component will not affect the equilibrium option price. Thus, we can not “act as if” the jump component was not there and compute the correct option price. These are emphasized many times by Merton [33, 34].

Appendix. To solve a heat equation

The partial differential equation (heat equation)
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\[
\begin{cases}
-\varphi_t(t, y) + \beta \varphi(t, y) - \alpha y \varphi_y(t, y) - \frac{1}{2} \theta^2 y^2 \varphi_{yy}(t, y) = g(t, y), \\
\varphi(T, y) = f(y), \quad t \in [0, T), \quad y \in \mathbb{R}, \quad \beta, \alpha, \theta \text{ are all constant},
\end{cases}
\]  

(1)

is often used in stochastic financial market without jump, and the famous Black-Scholes formula is derived from this kind of equation. Now let us solve this equation.

Let \( y = e^x \), then \( x = \ln y \) (if \( y < 0 \), then let \( y = -e^x \), all the conclusions are the same), and then

\[
\varphi_y = \frac{\partial \varphi}{\partial x} \cdot \frac{\partial x}{\partial y} = \frac{1}{y} \varphi_x = e^{-x} \varphi_x,
\]

\[
\varphi_{yy} = (\varphi_y)_y = e^{-x} (e^{-2x})_y = e^{-x} (-e^{-x} \varphi_x) = e^{-2x} (\varphi_{xx} - \varphi_x).
\]

Substitute into (1), we obtain

\[
-h(t, x) + \beta \varphi - \varphi_t - (\alpha - \frac{1}{2} \theta^2) \varphi_x = \frac{\theta^2}{2} \varphi_{xx},
\]

(2)

where \( h(t, x) = g(t, e^x) \). Make a transformation of \( \xi = x - \left(\alpha - \frac{\theta^2}{2}\right)t, \quad \tau = t \), then

\[
\varphi_t = \varphi_\xi \cdot \xi_t + \varphi_\tau \cdot \tau_t = - \left(\alpha - \frac{\theta^2}{2}\right) \varphi_\xi + \varphi_\tau,
\]

\[
\varphi_x = \varphi_\xi \cdot \xi_x + \varphi_\tau \cdot \tau_x = \varphi_\xi, \quad \varphi_{xx} = (\varphi_x)_\xi \cdot \xi_x + (\varphi_x)_\tau \cdot \tau_x = \varphi_{\xi\xi}.
\]

Substitute into (2), we arrive to

\[
-K(\tau, \xi) + \beta \varphi - \varphi_\tau = \frac{\theta^2}{2} \varphi_{\xi\xi},
\]

where \( K(\tau, \xi) = h \left(\tau, \xi + \left(\alpha - \frac{\theta^2}{2}\right)\tau\right) \). Multiplying two side of the previous equation by \( e^{-\beta \tau} \) leads to

\[
-K(\tau, \xi) \cdot e^{-\beta \tau} - (e^{-\beta \tau} \varphi)_\tau = \frac{\theta^2}{2} [e^{-\beta \tau} \varphi]_{\xi\xi}.
\]

Let \( \psi = e^{-\beta \tau} \varphi \), the equation becomes

\[
\begin{cases}
-K(\tau, \xi) \cdot e^{-\beta \tau} - \psi_\tau = \frac{\theta^2}{2} \psi_{\xi\xi}, \quad \xi \in \mathbb{R}, \quad \tau \in [0, T), \\
\psi(T, \xi) = e^{-\beta T} f(e^{\xi + (\alpha - \frac{\theta^2}{2})T}).
\end{cases}
\]  

(3)

It follows from [14] that the existence, uniqueness and stability for the solution of this heat equation are valid. And there is an explicit expression in [14] (P84) for the solution. After deriving the solution of (3), make inverses of above-mention transformations (Note: Finally, we should substitute back \( x = \ln |y| \) into the equation, because there is the possibility of \( y < 0 \)), and the solution of the original equation comes out.
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