RETHINKING VARIATION IN THE APPLICATION OF THE QUEUEING FORMULA: $L = \lambda W$

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ABSTRACT

The expression L=λW, also known as Little's Law has been used to describe a queuing model. However, the expression L=λW gives no consideration of variation over time and data homogeneity. The purpose of this paper is to examine process variation over time in the expression L=λW using Shewhart control charts.

Keywords: Little's Law, Queueing, Control Charts

Introduction

The expression $L = \lambda W$, also known as Little's Law or Theorem, has been used to describe a queuing model where L is the expected number of units in the system, W is the expected time spent by a unit in the system and $1/_{\lambda}$ is the expected time between two consecutive arrivals (hence λ is the expected arrival rate to the system). It has been shown that the results are free of any assumptions regarding the arrival and service distributions, independence of interarrival times, number of channels (servers, workstations) and queue discipline, hence, the theorem offers considerable generalization (Little, 1961). The requirement of steady state, that the state of the current state is independent of the starting state, is assumed where over a long period of time the average arrivals into the system will equal the average departures out of the system. The relationship has been covered in introductory operations research, operations management and management science textbooks and it has been used in solving many manufacturing, waiting line, inventory and other related problems.

In the proof of the theorem, random characteristics of interarrival and processing times distributions have been considered. Considering the arrivals and departures occur in sequence, there should be a way to monitor variation over time in interarrival times, arrivals and departures. Also, when computing mean values to substitute in the expression $L = \lambda W$, it is assumed that the data set is homogenous; that is, all values came from the same universe. However, current practices in the application of Little's Law, the expression $L = \lambda W$ gives no consideration of variation over time and data homogeneity. In the absence of such insights the process changes are missed, inappropriate decisions made, and resources wasted.

The purpose of this paper is 1) to examine process variation over time in the expression $L = \lambda W$ using Shewhart control chart, 2) to show that for a meaningful analysis of queuing model process data must be homogeneous, and 3) to propose the control chart as a test to monitor variation and for data set homogeneity. As a visual device, control chart can provide an effective method for understanding and monitoring process variation and facilitate economic decision making by characterizing the types of variation.

An extensive literature search resulted only one article, Shore (2006), where control charts have been used for the queue length in a generalized system. However, many of control chart assumptions set forth by Shewhart have not been considered in this paper.

Variation and Quality Control Charts

Shewhart control charts have been proven to be an effective way to improve industrial processes. One of Shewhart's greatest contributions is to characterize the sources of process variation into two categories: chance causes of variation and assignable causes of variation. The eminent statistician W. Edwards Deming expanded on the understanding of variation as the basis for management decision making (Shewhart, 1931; Deming; 1986, 1994). For pedagogical reasons, Deming replaced Shewhart's definitions with common causes of variation and special causes of variation. A process that exhibits variation from common causes is said to be in statistical control; a predictable process within limits. Conversely, a process that exhibits variation due to special causes is not considered in statistical control and is therefore unpredictable.

The distinction between the two sources of variation is paramount in process analysis, as they signify two different types of action. Variation due to common causes will require consideration of many sources and their interaction. When variation due to special cause is present, the action is very specific and focused on the special cause. The confusion of the two sources of variation can lead to two types of errors. Error I occurs when one reacts to a data value as if it came from a special cause when, in fact it came from common causes of variation; Error II occurs when one reacts to a data value as if it came from common causes of variation, when in fact it was due to a special cause. In order to distinguish between the two sources of variation and to facilitate the appropriate action, Shewhart provided the framework by establishing the three sigma limits as the boundaries to define the types of variation. Data values randomly behaving within the three sigma limits represent common cause variation, that is, a constant system of causes. Any value outside the three sigma limits is deemed to be caused by special cause(s) of variation. Therefore, the three sigma limits provide a balance to minimize the possibility of making errors I or II. The basis of the three sigma limits, Shewhart argued, are mathematical theory, empirical evidence and practical experience (Shewhart, 1931). Although Shewhart suggested a value beyond the three sigma limits as a signal of special causes of variation, others signals have been suggested (Western Electric Handbook, 1956; Wheeler & Chambers, 1992).

Materials and Methods

Individual and Moving Range control chart is most suitable when the data is collected periodically, e.g. daily or monthly. The method used to estimate the three-sigma limits, lower control limit (LCL) and upper control limit (UCL), for the individual values are $LCL = \bar{X} - 2.66\overline{mR}$ and $UCL = \bar{X} + 2.66\overline{mR}$ where $\overline{\mathrm{X}} = \frac{1}{\mathrm{n}}\sum \mathrm{x}$ is the average of individual values, and $\overline{\mathrm{mR}} = \frac{1}{\mathrm{n}-1}\sum \mathrm{mR}$ is the average moving range, the difference of two successive measurements. For the moving range chart, $UCL = 3.26\overline{mR}$ and lower control limit is set to zero. In cases where the average moving range is affected by very large moving range values, it is recommended to use median moving range ($\widetilde{mR}$). The control limits for individual and median moving range control chart is calculated as $\bar{X} \pm 3.14\widetilde{mR}$. For detailed study of individual and moving range control chart the reader is encouraged to review Wheeler and Chambers (1992) and Wheeler (2003). Two cases have been presented in this paper. The control charts are produced using MINITAB Release 14 (State College, PA).

Case I: Manufacturing Production

Table 1 shows the data from a manufacturing production process where annual orders have been converted to orders per day (λ) and average cycle time (W) is the number of days to manufacture an order. On average 0.037 orders arrive per day (13.5 orders annually); hence, mean interval between each order is $1/\lambda = 27$ days. Average cycle time to process an order is 25.6 days. Substituting the average values in the expression $L = \lambda W$ to determine the average number of orders in the system for the entire dataset, that is the queue length, will result in $L = 0.037 \times 25.6 = 0.95$ orders. Queue length for each period is also calculated and are shown in Table 1. Figures 1, 2 and 3 show the Individual and Moving Range control charts for orders per day (λ), average cycle time (W) and average number of order in the system (L) respectively.

Table 1: Daily Rates and Average Cycle Times for the Production Process

Period	Daily Rate (λ)	Average Cycle Time (W)	Queue Length (L)
1	0.041	34.7	1.42
2	0.036	31.1	1.12
3	0.044	27.1	1.19
4	0.044	27.2	1.20
5	0.038	25.3	0.96
6	0.036	22.5	0.81
7	0.033	23.1	0.76
8	0.033	22.0	0.73
9	0.041	21.1	0.87
10	0.027	21.5	0.58

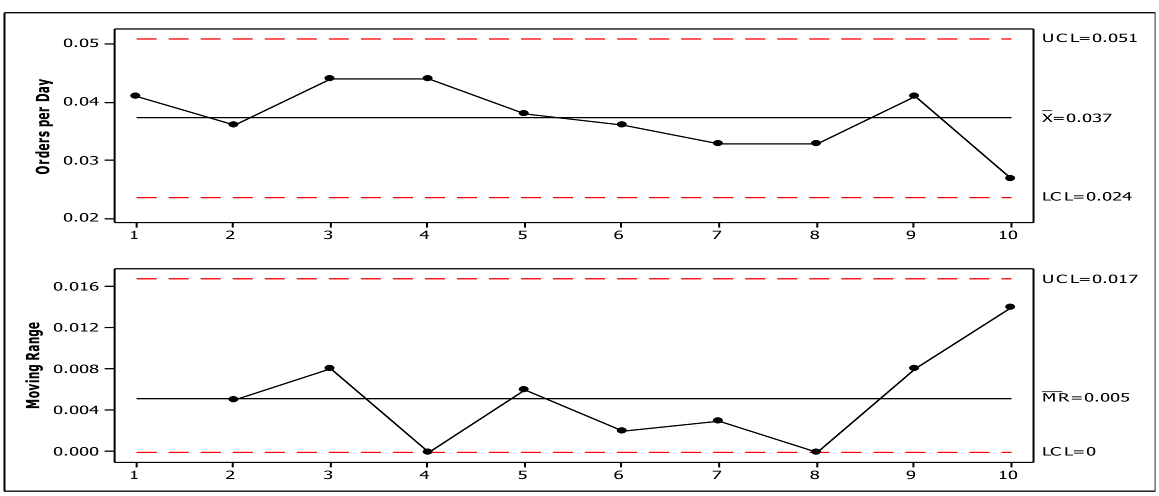

Figure 1: Individual and Moving Range Control Chart of Orders per Day (λ) for the Production Process

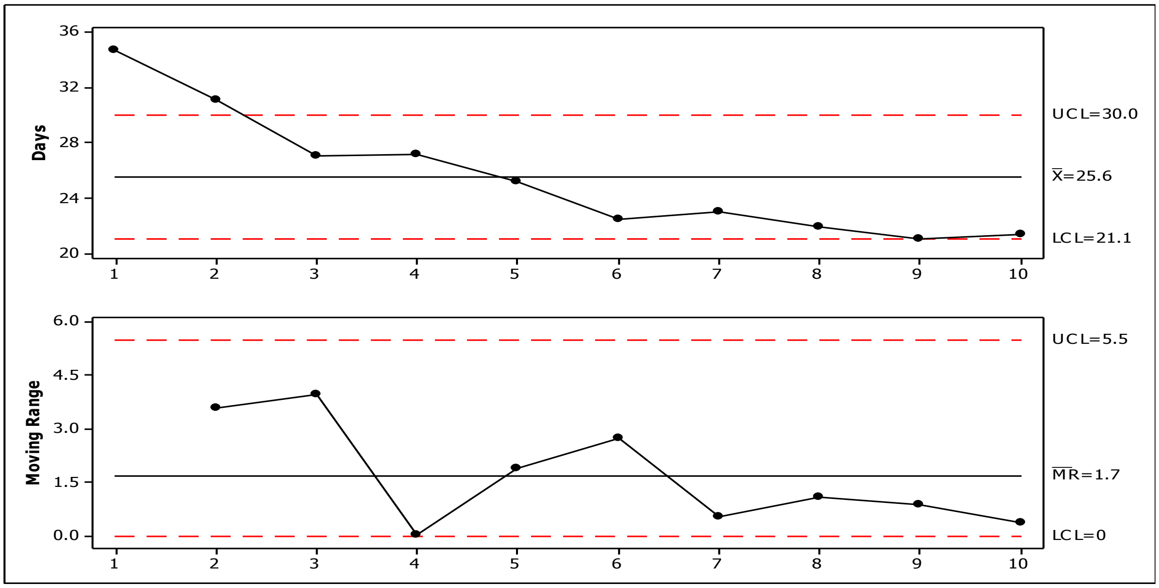

Figure 2: Individual and Moving Range Control Chart of Average Cycle Days per Order (W) for the Production Process

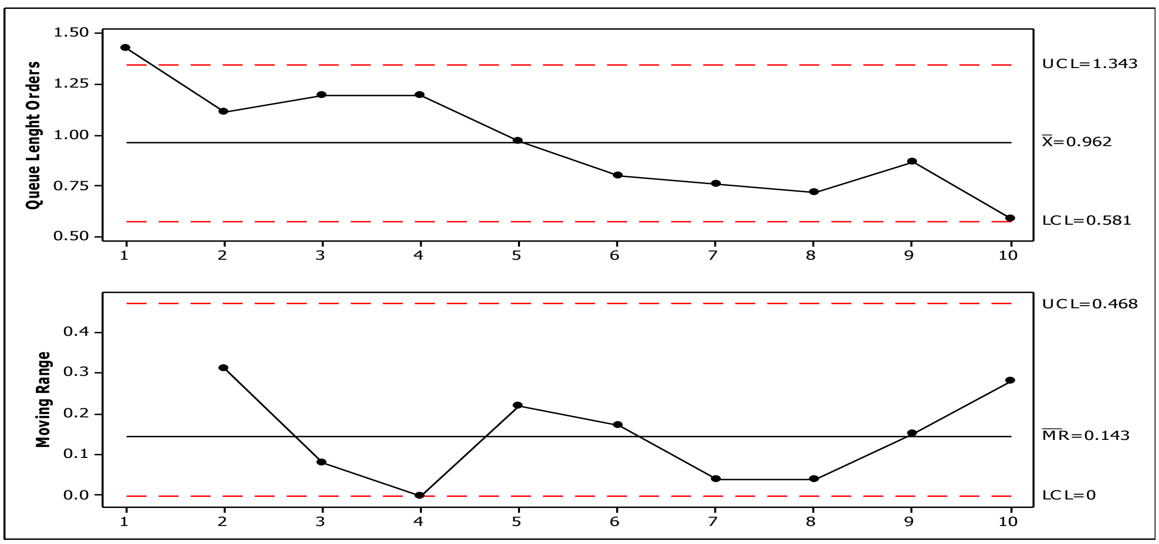

Figure 3: Individual and Moving Range Control Chart of Queue Length (L) for the Production Process

Case II: Plan Review and Approval

Case II is the data for the construction plan review and approval process for a governmental agency. Such data for one year were used for this case where, on average, 1.33 plans are submitted daily (λ) to the agency and, on average, it takes 54.1 days to review and approve a plan (W). Using Little's Law expression, the number of plans in the system is estimated as $L = 1.33 \times 54.1 \cong 72$ plans. Figure 3 and 4 show the Individual and Median Moving Range control charts for daily submission rate (λ) and review and approval time (W) respectively.

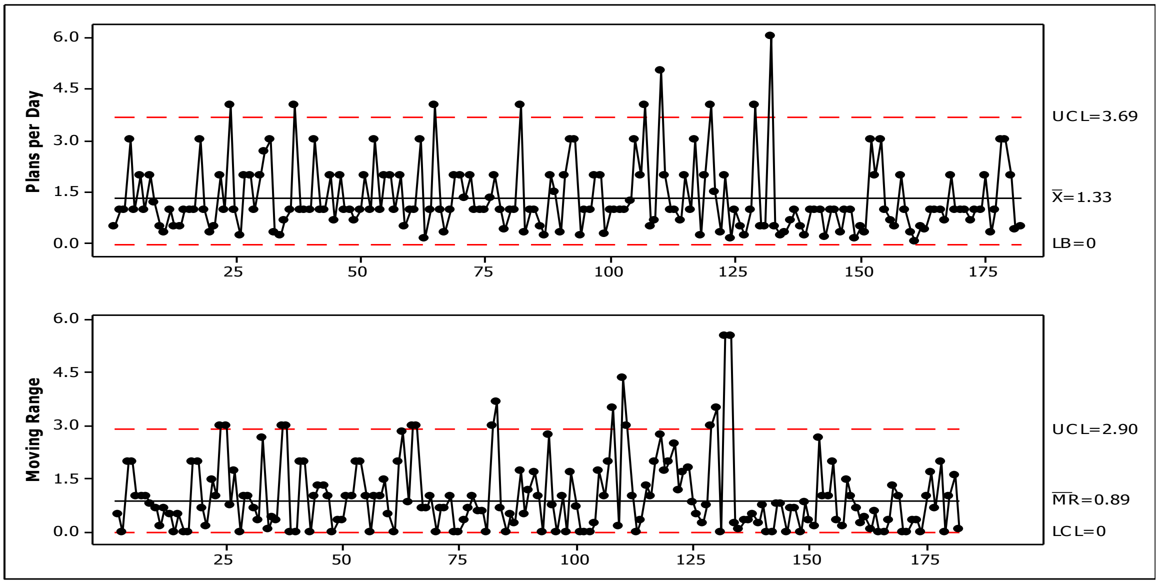

Figure 4: Individual and Median Moving Range Chart for Plans per Day Rate (λ)

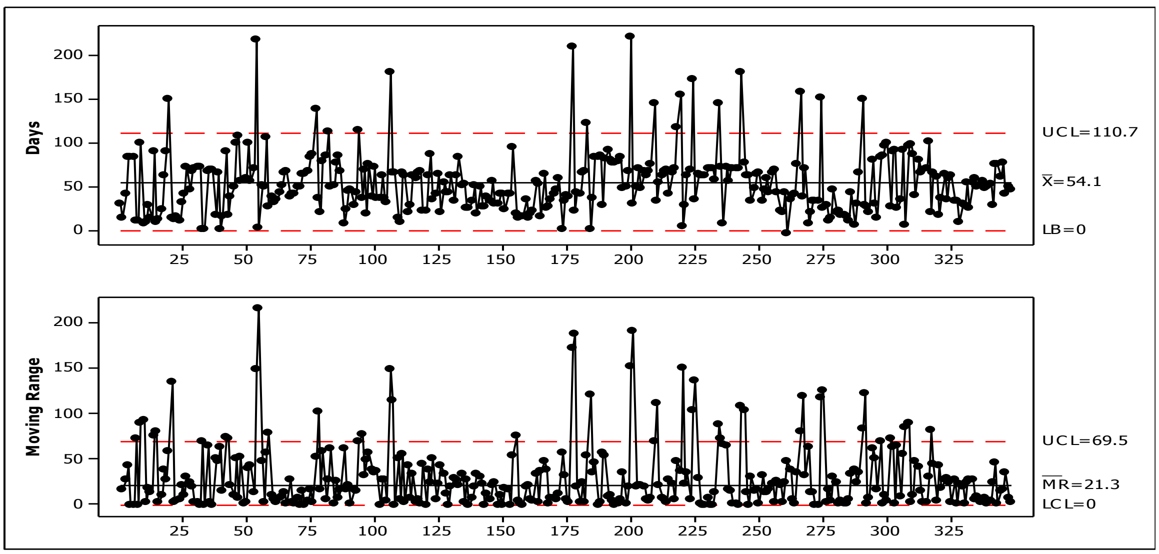

Figure 5: Individual and Median Moving Range Chart for Plan Approval Days (W)

Discussion

When computing statistics for process analysis, measurements must be homogenous, that is, the data must come from the same universe. A meaningful description of performance measures such as wait and processing times, arrival rate or interarrival time cannot be expressed merely as mean values without considering the variability, as the mean is of very little help without knowledge of variation about the mean. One can observe the variation about the mean by simply plotting the data in the chronological order. Then, the variation can be characterized either as common cause or special cause using Shewhart control chart. In the case of common cause variation, Shewhart (1931, 1986) framed a constant system of causes implying that the data came from the same universe, the condition for homogeneity. Several authors have suggested using control chart for a test of data homogeneity; one can assume the data represent a homogenous system when the variation over time is characterized as common cause (Mandel, 1964; Juran 1974; Duncan 1986; Grant and Leavenworth, 1988; Wheeler, 1995, 2003). Under the condition of homogeneity, the dataset displays consistency and the process is predictable in the near future; therefore, the average and the control limits will provide the process capability. Conversely, in the presence of special causes, that is, an unstable process, the data presents a nonhomogenous system and therefore the use of average is meaningless. The latter process has no predictive value.

In the first case, Figure 1, Individual and Moving Range control chart of daily order rate (λ), shows a process that is in statistical control. Lower control limit of 0.024 orders per day (8.8 orders annually) and upper control limit of 0.051 orders per day (18.6 orders annually) suggest the process capability; that is, based on previous experience the orders will fluctuate between approximately 9 and 19 orders annually. This is the range of prediction for orders.

Figure 2 shows the Individual and Moving Range control chart of average cycle days (W) for the same time period. The average cycle time is 25.6 days, lower control limit of 21.1 and upper control limit of 30.0 days. However, the chart displays special causes, two values above upper control limit; and a sequence of four values above and a sequence of six values below the average line. Figure 3 shows the Individual and Moving Range control chart of the queue length (L). This chart also exhibits special cause, one value above the upper control limit; and a run of five values above and a run of five values below the average line. Hence the queue length exhibits two systems of causes, the first five queue lengths producing an average of 1.2 orders and the remaining five values producing an average of 0.75. Using the overall average queue length of $L = 0.95$ orders assumes homogeneity and it is misleading.

In the second case, Figure 4, Individual and Median Moving Range control chart of plan submission rate (λ), shows special cause variation with nine rates above the upper control limit and three long runs below the average line. Figure 5, Individual and Median Moving Range control chart of approval days (W), also shows special cause variation with eighteen values above the upper control limit and four long runs below the average line. According to Shewhart criterion, both charts exhibit processes that are unstable and unpredictable, therefore, represent nonhomogenoues systems. Therefore, assuming $\lambda = 1.33$ and $W = 54.1$ to estimate the queue length ($L = 1.33 \times 54.1 \cong 72$ orders) does not represent the real process changes. Calculating average rate and time from a system that is changing will mask real process changes.

Although the arrivals and departures occur in sequence, the sequence is masked when only the mean values are reported. Considering that the process changes over time, there should be a way to monitor the changes and characterize the variation. A simple time series where values are shown in order would be the first appropriate approach to process analysis. Second, a quality control chart can characterize the variation as common cause or special cause. If the data set manifests common causes only, the process capability is known and therefore the process is predictable in the near future within the lower and upper control limits; homogeneity is assumed. Then perhaps expression $L = \lambda W$ can give a snap shot view of the queuing model. The assumption of steady state and use of expected value in the application of Little's Law without considering variation hinders true process analysis.

When one applies the expression $L = \lambda W$ an assumption of data homogeneity is made, either implicitly or explicitly, without any evidence. As the cases presented here, performance measures such as mean cycle time in Case I and arrival rate and processing time in Case II exhibit special causes; therefore, do not present a homogenous data sets. The parameters used represent systems that are changing; hence, the average values are not appropriate to use.

The aim here was to show the limitations of using expression $L = \lambda W$ in analyzing queue models. Perhaps this paper will encourage further research and investigation on topics such as process capacity, capacity utilization, scheduling and other related models from the viewpoint of variation and control charts.

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