ON THE ASYMPTOTIC OPTIMALITY OF SCHEDULING HEURISTICS

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ABSTRACT

Many scheduling heuristics have been proven to be so-called "asymptotically optimal" when the number of jobs approaches infinity. We observe, however, that such heuristics may not become more frequently optimal with an increase of the number of jobs. We empirically reveal that the asymptotic optimality rate of a certain "almost surely asymptotically optimal" heuristic may be zero. On the other hand, the optimality rates of some more advanced heuristics are very high and improve as the number of jobs increases. Furthermore, their asymptotic optimality rate seems to be one for broad families of probability distributions from which instances are randomly generated. Our findings justify the need for a revised definition of the heuristic asymptotic optimality.

Keywords: Scheduling, heuristics, computational analysis, simulation, optimality rate

INTRODUCTION

Without any loss of generality consider an intractable minimization scheduling problem involving *n* jobs. Let T^* be the minimum value of the objective function and let T^H denote the value of the objective function of a feasible solution yielded by a heuristic (an approximation algorithm) *H* for solving this problem. The performance of *H* has been examined with respect to its relative error $E^H = (T^H - T^*)/T^*$. Assuming that the problem instances are randomly generated, the relative error of many scheduling heuristics can be proven to converge to zero, as *n* increases to infinity. This convergence has been established in the sense of *in distribution*, *in expectation*, *in probability*, or *almost surely*, that is, according to the well-known definitions of convergence of random variables. A heuristic whose relative error converges to zero has been called "asymptotically optimal". In particular, *H* has been regarded as "almost surely asymptotically optimal" if $\Pr(\lim_{n\to\infty} E^H = 0) = 1$.

In this paper the asymptotic behaviour of *H* is examined with respect to the probability $\Pr(T^H = T^*)$ = $\Pr(E^H = 0)$, which is called the optimality rate of *H*. We observe that some heuristics may satisfy $\lim_{n\to\infty} \Pr(T^H = T^*) = 1$ for a broad family of probability distributions from which problem instances are randomly generated. On the other hand, $\lim_{n\to\infty}\Pr(T^H = T^*) = 0$ may hold for heuristics thus far regarded as "asymptotically optimal" even in the strongest sense of *almost surely*. Our findings justify the need for a revised definition of the heuristic asymptotic optimality. Clearly, in our opinion, a heuristic H should be called asymptotically optimal only if its asymptotic optimality rate is 1.

We empirically examine the optimality rate of heuristics for solving the following scheduling problem. Each of n jobs available at time zero has to be processed on one machine. A job j is characterized by a release time r_j, when it becomes available for processing, processing time p_j and a subsequent delivery time q_j. Each job must be processed on the machine sometime after its release time, and its delivery begins immediately after processing has been completed. No job preemptions are allowed, at most one job can be processed at a time, but all jobs may be simultaneously delivered. The problem, commonly referred to as $1|r_j,q_j|C_{\max}$, is to determine a schedule that minimizes the time by which all jobs are delivered. Although $1|r_j,q_j|C_{\max}$ was shown to be strongly NP-hard by (Lenstra et al., 1977), it has been believed to be somewhat "easy"; see e.g. (Carlier, 1982; Grabowski et al.,1986; Hall and Shmoys, 1992; Mastrolilli and Bianchi, 2005; Gharbi and Labidi, 2010).

The symmetric problems $1|r_j|C_{\max}$ and $1|q_j|C_{\max}$ can be optimally solved by a scheduling rule of (Jackson, 1955). In the first (second) case, the jobs should be sequenced by their non-decreasing (non-increasing) release times (delivery times). For the $1|r_j,q_j|C_{\max}$ problem, Jackson's rule can be extended by scheduling first a job with the longest delivery time among all available jobs. The heuristic, say J, that employs the extended Jackson's rule can be implemented in $O(n\log n)$, and satisfies $T^J \leq [2 - 3/(L+1)]T^*$, where L is the load of the machine; see (Kise et al., 1979). (Kaminsky, 2003) proved that, under certain mild restrictions imposed on the probability distributions from which the job processing, release and delivery times are randomly and independently drawn, the relative error of J almost surely converges to zero as n approaches infinity. We notice, however, that for some examples of such distributions, the asymptotic optimality rate of J, $\lim_{n\to\infty}\Pr(T^J = T^*)$, is zero.

(Potts, 1980) presented a 3/2-heuristic for solving $1|r_j,q_j|C_{\max}$, that is, a heuristic with the worst-case performance ratio of 3/2. This ratio was later reduced to 4/3 by (Hall and Shmoys, 1992). The $O(n^2\log n)$ heuristics of Potts (P) and Hall and Shmoys (HS) run J up to n and $8(n-1)$ times, respectively. (Nowicki and Smutnicki, 1994) developed a 3/2-heuristic, which requires only $O(n\log n)$ time.

In this paper we empirically examine the performance of $1|r_j,q_j|C_{\max}$ heuristics with respect to their optimality rates. Since T^* cannot be easily determined, these rates are underestimated by the probabilities $\Pr(T^H = LB)$, where LB is a very effective $O(n\log n)$ lower bound on T^*. In particular, we observe that for $H = P$ and HS, $\Pr(T^H = LB)$ is very high and it increases with an increase of n. Thus, the strongly NP-hard $1|r_j,q_j|C_{\max}$ problem appears to be easier to solve optimally when its size n becomes larger. Furthermore, the obtained results allow us to conjecture that $\lim_{n\to\infty}\Pr(T^H = LB) = 1$, whenever job processing times are randomly and independently drawn from a common distribution belonging to a broad family of probability distributions, and the same refers to release and delivery times.

The remainder of the paper is organized as follows. In Section 2 we recall basic facts concerning the $1|r_j,q_j|C_{\max}$ problem and define an effective lower bound on T^*. The analyzed heuristics for solving the problem are described in Section 3. The results of our extensive simulation experiments on the optimality rates of $1|r_j,q_j|C_{\max}$ heuristics are presented in Section 4. Section 5 includes a brief summary of the obtained results.

BACKGROUND

Let every job $j = 1,2,...,n$ be described by its non-negative release, processing and delivery times denoted by r_j, p_j and q_j, respectively. If $p_j = 0$, job j must visit the machine for an instant. The delivery completion time of the schedule yielded by a given job sequence $\pi = (\pi(1),\pi(2),...,\pi(n))$ is expressed by:

$$T(\pi) = \max_{1\le i\le k\le n} [r_{\pi(i)} + \sum_{j=i}^{k} p_{\pi(j)} + q_{\pi(k)}], \tag{1}$$

and, therefore,

$$T(\pi) = r_{\pi(i^*)} + \sum_{j=i^*}^{k^*} p_{\pi(j)} + q_{\pi(k^*)}, \tag{2}$$

for some jobs $\pi(i^*)$ and $\pi(k^*)$ with $i^* \le k^*$. Since multiple pairs $(\pi(i^*),\pi(k^*))$ may exist, we assume as (Potts, 1980) that both i^* and k^* are as small as possible. All jobs $\pi(j)$ for which $i^* \le j \le k^*$ are called critical in π, and they can be found, together with $T(\pi)$, in $O(n)$ time by the critical path method.

The $1|r_j,q_j|C_{\max}$ problem is to find a job sequence π^* that minimizes $T(\pi)$; for short $T^* = T(\pi^*)$. Since the problem is symmetric with respect to r_j and q_j, one can consider the inverse problem defined by switching r_j and q_j for all jobs.

In order to successfully underestimate the optimality rates of $1|r_j,q_j|C_{\max}$ heuristics, one needs a very effective lower bound on T^*. Rather surprisingly, such a lower bound, LB, is the shortest delivery completion time of the $1|r_j,q_j,pmnt|C_{\max}$ problem, that is, $1|r_j,q_j|C_{\max}$ with job preemptions allowed. LB can be computed in $O(n\log n)$ time by using the extended Jackson's rule; see (Horn, 1974). The results presented in Section 4 allow us to conjecture that $\lim_{n\to\infty} \Pr(LB = T^*) = 1$, whenever job processing times are randomly and independently drawn from a distribution belonging to a broad family of probability distributions, and the same also refers to release and delivery times. This indicates that job preemptions are actually *useless* for very large instances of $1|r_j,q_j,pmnt|C_{\max}$.

HEURISTICS

J Heuristic (The Extended Rule Of (Jackson, 1955), The Algorithm Of (Schrage, 1971))

Sequence the jobs according to the following rule: at any time select a job with the longest delivery time among all jobs available to be processed.

Let $\pi^J = (\pi^J(1),\pi^J(2),...,\pi^J(n))$ be a sequence produced by *J*, and $\pi^J(i^*)$ and $\pi^J(k^*)$ the first and last critical jobs in π^J. It is well known that if for every critical job $\pi^J(j)$, $q_{\pi^J(j)} \geq q_{\pi^J(k^*)}$ holds, then π^J is optimal. Otherwise, (Potts, 1980) proposed to find the last critical job, say $\pi^J(j^*)$, for which $q_{\pi^J(j^*)} < q_{\pi^J(k^*)}$, and delay it by replacing its release time with $r_{\pi^J(k^*)}$; he called $\pi^J(j^*)$ the interference job.

P_I Heuristic Of (Potts, 1980)

repeat

run *J* on the current instance to find π^J, and locate the first and last critical jobs $\pi^J(i^*)$ and $\pi^J(k^*)$ in π^J; **if** the interference job $\pi^J(j^*)$ exists **then** set $r_{\pi^J(j^*)} := r_{\pi^J(k^*)}$

until either there is no interference job or *I* runs have been performed

return the shortest π^J generated.

(Hall and Shmoys, 1992) noticed that, with one exception, the 3/2 worst case performance ratio of the $P = P_n$ heuristic of Potts can be reduced to 4/3 when *P* is run separately on both the original and inverse problems. The exception involves the case when there are two jobs *u* and *v* with $p_u > L/3$ and $p_v > L/3$, where $L = \sum_{j=1}^{n} p_j$. Since in our simulation experiments this theoretical case is unlikely to occur, a detailed description of how *HS* deals with it is omitted; *HS* needs then up to 8(*n*-1) runs of *J*.

ESTIMATING OPTIMALITY RATES

Let T^H be the delivery completion time of the schedule produced by a heuristic *H*. We are interested in the relative error $E^H = (T^H - T^*)/T^*$ induced by *H*. In particular, we will examine the optimality rate of *H*, that is, $\Pr(T^H = T^*) = \Pr(E^H = 0)$. Since the optimal makespan T^* cannot be easily determined, we overestimate $(T^H - T^*)/T^*$ and underestimate $\Pr(T^H = T^*)$ by replacing T^* with the lower bound *LB* introduced in Section 2. Consequently, $E^H \leq (T^H - LB)/LB$ and $\Pr(T^H = T^*) \geq \Pr(T^H = LB)$.

In our first simulation experiment, job processing times p_j were generated from the uniform distribution over the integers 0,1,...,100, denoted by U({0,1,...,100}), release times r_j from

U({0,1,...,100Rn}), while delivery times q_j from U({0,1,...,100Qn}), where the multipliers R,Q are 0.1, 0.5 or 1; see (Kaminsky, 2003; Haouari and Ladhari, 2007). For every n = 20, 40, 60, 80, 100, 150, 200, 500, 1000, and 9 pairs (R,Q), we generated N = 2000 instances for which we computed LB and T^H to estimate Pr(T^H = LB) and the maximum relative deviation denoted for short by Max[(T^H – LB)/LB]. The obtained estimates for H = HS and P are shown in Tables 1 and 2. Note that, for example, the P result of 0.9970 in Table 1, obtained for n = 1000 and (R,Q) = (0.1,0.1), means that we could not verify P's optimality for six out of N = 2000 generated instances.

TABLE 1. THE ESTIMATED PR(T^H = LB); THE CASE OF THE U({0,1,...,100}) DISTRIBUTION.

R	Q	Heuristic	n=20	n=40	n=60	n=80	n=100	n=150	n=200	n=500	n=1000
0.1	0.1	P	0.9400	0.9650	0.9790	0.9855	0.9875	0.9865	0.9915	0.9965	0.9970
		HS	0.9400	0.9650	0.9790	0.9855	0.9875	0.9865	0.9915	0.9965	0.9970
	0.5	P	0.6235	0.6950	0.7760	0.8285	0.8690	0.9170	0.9465	0.9760	0.9890
		HS	0.6600	0.7585	0.8235	0.8740	0.9080	0.9490	0.9645	0.9805	0.9915
	1	P	0.7680	0.8425	0.8950	0.9195	0.9480	0.9635	0.9780	0.9895	0.9965
		HS	0.8130	0.8825	0.9325	0.9495	0.9650	0.9765	0.9860	0.9905	0.9970
0.5	0.1	P	0.6255	0.7045	0.7990	0.8530	0.8720	0.9190	0.9430	0.9775	0.9925
		HS	0.6645	0.7540	0.8430	0.8915	0.9045	0.9435	0.9580	0.9835	0.9955
	0.5	P	0.8735	0.9430	0.9675	0.9730	0.9780	0.9865	0.9900	0.9960	0.9995
		HS	0.8995	0.9560	0.9775	0.9875	0.9870	0.9925	0.9925	0.9985	1.0000
	1	P	0.9525	0.9750	0.9815	0.9870	0.9915	0.9955	0.9970	0.9980	1.0000
		HS	0.9635	0.9800	0.9870	0.9930	0.9955	0.9960	0.9980	0.9985	1.0000
1	0.1	P	0.7930	0.8690	0.9250	0.9470	0.9565	0.9645	0.9765	0.9925	0.9950
		HS	0.7985	0.8815	0.9315	0.9545	0.9620	0.9705	0.9810	0.9930	0.9960
	0.5	P	0.9525	0.9800	0.9885	0.9915	0.9925	0.9955	0.9960	0.9995	0.9990
		HS	0.9620	0.9825	0.9905	0.9920	0.9940	0.9970	0.9970	0.9995	0.9990
	1	P	0.9820	0.9900	0.9930	0.9970	0.9985	0.9980	0.9985	1.0000	1.0000
		HS	0.9850	0.9925	0.9955	0.9985	0.9990	0.9980	0.9990	1.0000	1.0000

One can conclude that the optimality rates of P and HS are very high, and their asymptotic optimality rate is most likely equal to 1 for all examined pairs (R,Q). On the other hand, the maximum relative errors are very small and evidently converge to 0 as n increases. Since P is much faster than HS, in the sequel we present only the Pr(T^P = LB) results. Note here that Pr(T^P = LB) ≤ Pr(T^{HS} = LB) because the schedule yielded by HS is never worse than that of P.

We repeated the above experiment by generating times p_j, r_j and q_j from U([0,1]), U([0,Rn]) and U([0,Qn]), respectively. The obtained underestimates of the optimality rates of P are presented in Table 3. As one could expect, the results are slightly worse than those given in Table 1. Thus, discreteness leads to higher optimality rates. We also analyzed the case of the U({0,1,...,10}), U({0,1,...,10Rn}) and U({0,1,...,10Qn}) distributions for p_j, r_j and q_j, respectively, to confirm that lower discreteness leads to better rates; the results are available upon a request.

TABLE 2. THE ESTIMATED MAX[(T^H – LB)/LB)]; THE CASE OF THE U({0,1,...,100}) DISTRIBUTION.

R	Q	Heuristic	n=20	n=40	n=60	n=80	n=100	n=150	n=200	n=500	n=1000
0.1	0.1	P	0.03411	0.00903	0.00593	0.00536	0.00410	0.00225	0.00240	0.00044	0.00012
		HS	0.03411	0.00903	0.00593	0.00536	0.00410	0.00225	0.00240	0.00044	0.00012
	1	P	0.03460	0.01660	0.01087	0.00739	0.00724	0.00390	0.00313	0.00124	0.00030
		HS	0.03213	0.01504	0.00758	0.00453	0.00346	0.00194	0.00152	0.00081	0.00025
	0.5	P	0.05059	0.03407	0.02162	0.01356	0.01381	0.00836	0.00582	0.00229	0.00061
		HS	0.04992	0.02703	0.01656	0.01156	0.01040	0.00459	0.00363	0.00112	0.00043
1	0.1	P	0.02453	0.01611	0.01045	0.00597	0.00568	0.00346	0.00251	0.00070	0.00032
		HS	0.02124	0.01166	0.00840	0.00460	0.00428	0.00262	0.00173	0.00046	0.00018
	1	P	0.01686	0.00827	0.00344	0.00241	0.00031	0.00077	0.00108	0.00000	0.00000
		HS	0.00917	0.00532	0.00337	0.00121	0.00031	0.00077	0.00030	0.00000	0.00000
	0.5	P	0.01663	0.00815	0.00459	0.00366	0.00454	0.00179	0.00134	0.00027	0.00006
		HS	0.01413	0.00649	0.00412	0.00318	0.00111	0.00179	0.00054	0.00027	0.00005
0.5	0.1	P	0.05018	0.03650	0.01970	0.01385	0.01346	0.00810	0.00485	0.00207	0.00101
		HS	0.04213	0.02578	0.01419	0.01194	0.01044	0.00507	0.00318	0.00112	0.00018
	1	P	0.02238	0.01172	0.00656	0.00524	0.00293	0.00345	0.00080	0.00038	0.00000
		HS	0.01873	0.00610	0.00419	0.00218	0.00118	0.00126	0.00040	0.00023	0.00000
	0.5	P	0.04573	0.01831	0.00821	0.01256	0.00721	0.00526	0.00279	0.00069	0.00014
		HS	0.03246	0.01042	0.00620	0.00490	0.00447	0.00205	0.00199	0.00034	0.00000

TABLE 3. THE ESTIMATED PR(T^P = LB); THE CASE OF THE U([0,1]) DISTRIBUTION.

R	Q	n=20	n=40	n=60	n=80	n=100	n=150	n=200	n=500	n=1000
0.1	0.1	0.9305	0.9615	0.9770	0.9850	0.9865	0.9855	0.9900	0.9960	0.9970
	0.5	0.6135	0.6860	0.7690	0.8225	0.8635	0.9155	0.9410	0.9735	0.9880
	1	0.7625	0.8365	0.8940	0.9170	0.9485	0.9635	0.9760	0.9880	0.9960
0.5	0.1	0.6155	0.6945	0.7920	0.8475	0.8695	0.9175	0.9430	0.9745	0.9900
	0.5	0.8745	0.9430	0.9675	0.9730	0.9780	0.9860	0.9905	0.9960	1.0000
	1	0.9495	0.9750	0.9825	0.9865	0.9905	0.9950	0.9970	0.9980	1.0000
1	0.1	0.7885	0.8670	0.9220	0.9450	0.9560	0.9635	0.9765	0.9920	0.9950
	0.5	0.9505	0.9795	0.9880	0.9910	0.9915	0.9955	0.9960	0.9995	0.9990
	1	0.9805	0.9900	0.9930	0.9970	0.9980	0.9980	0.9985	1.0000	1.0000

In order to examine the impact of the type and skewness of probability distributions from which instances are generated we conducted two additional experiments. Firstly, we considered some binomial distributions $B(\hat{n}, p)$ with $\hat{n}$ trials and a probability p of success per trial. We assumed $\hat{n}$ = 100 (to secure a comparison with Table 1 results), and p = 0.25, 0.5 and 0.75. Hence, times p_j, r_j and q_j were generated from B(100,p), B(100Rn,p) and B(100Qn,p), respectively. Secondly, we considered Weibull distribution W(α,μ) with mean μ = 1 and shape parameter α = 1/2, 1 and 2. Thus, p_j, r_j and q_j were generated from W(α,1), W(α,Rn) and W(α,Qn). The obtained $\Pr(T^P = LB)$ results restricted to n = 500 and 1000 are shown in Table 4.

TABLE 4. THE ESTIMATED PR(T^P = *LB*) FOR *N* = 500, 1000; THE CASES OF B(100,*P*) AND W(A,1) DISTRIBUTIONS.

		Binomial Distributions						Weibull Distributions					
		B(100,0.25)		B(100,0.5)		B(100,0.75)		W(0.5,1)		W(1,1)		W(2,1)	
R	Q	n=500	n=1000	n=500	n=1000	n=500	n=1000	n=500	n=1000	n=500	n=1000	n=500	n=1000
0.1	0.1	0.9985	0.9995	0.9980	0.9985	0.9985	0.9990	1.0000	1.0000	0.9975	0.9995	0.9935	0.9935
	0.5	0.9980	0.9990	0.9980	0.9995	0.9975	0.9980	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0	0.9975	0.9995	0.9965	0.9990	0.9970	0.9990	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	0.1	0.9980	0.9980	0.9975	0.9995	0.9965	0.9990	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995
	0.5	0.9960	0.9980	0.9995	0.9990	0.9985	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0	0.9965	0.9970	0.9950	0.9985	0.9965	0.9995	1.0000	0.9995	1.0000	1.0000	1.0000	1.0000
1.0	0.1	0.9985	0.9980	0.9975	0.9975	0.9965	0.9985	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.5	0.9960	0.9990	0.9970	1.0000	0.9955	0.9990	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0	0.9940	0.9975	0.9965	0.9990	0.9965	0.9985	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

By analyzing the performance of other heuristics, we noticed that for the pairs (*R*,*Q*) = (0.1,0.5) and (0.1,1), and the examined uniform and Weibull distributions from which p_j, r_j and q_j were generated, the asymptotic optimality rate of *J* seems to be zero. On the other hand, for these probability distributions, the relative error $E^J = (T^J - T^*)/T^*$ has been proven to satisfy $\Pr(\lim_{n\to\infty} E^J = 0) = 1$, that is, E^J almost surely converges to zero as *n* increases to infinity; see (Kaminsky, 2003). Table 5 presents the obtained estimates of probabilities Pr(T^J = *LB*) and maximum relative deviations Max[(T^J – *LB*)/*LB*] for (*R*,*Q*) = (0.1,1). Note that very high estimates of Pr(T^H = *LB*) given in Tables 1, 3 and 4 indicate that Pr(*LB* = T^*) is close to 1. Thus, the results shown in Table 5 reveal a conceptual difference between the concept of the asymptotic optimality rate and that of "almost sure asymptotic optimality". Clearly, since $\lim_{n\to\infty} \Pr(T^H = T^*) = 1$ implies $\Pr(\lim_{n\to\infty} E^H = 0) = 1$, but not conversely, in our opinion, a heuristic *H* can be called asymptotically optimal only if $\lim_{n\to\infty} \Pr(T^H = T^*) = 1$. Thus, by no means should *J* be called "asymptotically optimal", as it has been done so far.

TABLE 5. THE ESTIMATED PERFORMANCE MEASURES OF THE *J* HEURISTIC; THE CASE OF (*R*,*Q*) = (0.1,1).

Distribution	Performance Measure	n=20	n=40	n=60	n=80	n=100	n=150	n=200	n=500	n=1000
U({0,1,...,100})	Pr(T^J = *LB*)	0.0410	0.0200	0.0165	0.0180	0.0160	0.0110	0.0225	0.0195	0.0190
	Max[(T^J - *LB*)/*LB*]	0.0503	0.0242	0.0155	0.0120	0.0095	0.0060	0.0046	0.0018	0.0009
U([0,1])	Pr(T^J = *LB*)	0.0270	0.0030	0.0000	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000
	Max[(T^J - *LB*)/*LB*]	0.0515	0.0231	0.0153	0.0117	0.0092	0.0062	0.0046	0.0018	0.0009
W(0.5,1)	Pr(T^J = *LB*)	0.0875	0.0560	0.0450	0.0305	0.0335	0.0295	0.0210	0.0160	0.0150
	Max[(T^J - *LB*)/*LB*]	0.5858	0.4262	0.1066	0.0644	0.0506	0.0181	0.0130	0.0059	0.0017
W(1,1)	Pr(T^J = *LB*)	0.0405	0.0250	0.0210	0.0150	0.0150	0.0095	0.0065	0.0020	0.0005
	Max[(T^J - *LB*)/*LB*]	0.1525	0.0530	0.0401	0.0249	0.0218	0.0123	0.0078	0.0022	0.0010
W(2,1)	Pr(T^J = *LB*)	0.0380	0.0205	0.0190	0.0135	0.0115	0.0105	0.0075	0.0040	0.0020
	Max[(T^J - *LB*)/*LB*]	0.0662	0.0367	0.0226	0.0179	0.0125	0.0079	0.0054	0.0021	0.0008

It should be observed here that the almost sure convergence of E^J to zero has been analytically proven by the use of a weak lower bound on T^*:

$$LB0 = \min_{j\in J} r_j + \sum_{j\in J} p_j + \min_{j\in J} q_j, \quad (3)$$

for which $\lim_{n\to\infty} \Pr(LB0 = T^*) = 0$ and, consequently, $\lim_{n\to\infty} \Pr(T^H = LB0) = 0$ for every heuristic H. Thus, the bound $LB0$ is sufficient to prove the almost sure convergence of E^J to zero, but it becomes useless in underestimating the optimality rates of $1|r_j,q_j|C_{\max}$ heuristics. On the other hand, we strongly believe that the complexity of LB (as well as the complexity of T^H) excludes the chance of any analytical derivations concerning the heuristic optimality rates. Therefore, only empirical studies, as those conducted in this paper, can be successful in examining these rates.

The *HS* heuristic can be regarded as an extension of the *double J* (*DJ*) heuristic defined by running *J* on the original and inverse problems, and choosing a shorter sequence. The (3/2)-heuristic of (Nowicki and Smutnicki, 1994), say *NS*, is outperformed by *DJ*. This is rather surprising because the CPU times of *NS* and *DJ* are nearly the same, while the worst-case performance ratio of *DJ* is merely 2; see (Kise et al., 1979). This confirms the fact that worst-case performance ratios do not reflect the average performance of heuristics.

SUMMARY

The performance of heuristics for solving intractable optimization problems can be successfully evaluated by examining their optimality rate, that is, the probability of being optimal. In this paper we concentrated on the strongly NP-hard $1|r_j,q_j|C_{\max}$ problem, while $Fm|prmu|C_{\max}$ and $F2|r_j|C_{\max}$ were analyzed in (Kalczynski and Kamburowski, 2009; 2012). Our study revealed the need for a revision of the definition of the "heuristic asymptotic optimality". We showed that some simple heuristics, regarded so far as "almost surely asymptotically optimal", might have asymptotic optimality rate of zero. On the other hand, the asymptotic optimality rate of some more advanced heuristics can be 1 for broad families of probability distributions from which job characteristics are randomly and independently drawn. Therefore, we strongly believe that a heuristic can be called asymptotically optimal only if its asymptotic optimality rate is 1.

We underestimated the heuristic optimality rates by using lower bounds that had a very high optimality rate converging fast to 1 with an increase of the number of jobs. The existence of such bounds is crucial for conducting any studies on the heuristic optimality rates. Moreover, in our opinion, these studies can only be conducted empirically because any analytical proof in this matter is highly unlikely due to the lack of appropriate mathematical tools.

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