WILSON’S FORMULA WITH NONLINEAR HOLDING COST AND RANDOM QUALITY

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ABSTRACT

The classical Wilson’s square root formula is based on a series of highly restrictive assumptions among which are deterministic demand, constant holding costs across time and perfect quality of replenishment items. Many variants of the traditional Wilson model have been developed as a result of relaxing one or more of these assumptions. These variants include a class of Economic Order Quantity (EOQ) or Economic Manufacturing Quantity (EMQ) models that treat holding cost as a function of the amount of time that an item is held in inventory. Such models are known as perishable inventory models. A second class of EOQ and EMQ models that have received significant attention in the literature over the past few decades relax the perfect quality assumption by treating the number of defective units in each lot as a random variable. In this paper we utilize the basic framework of Wilson’s model with nonlinear holding cost which has appeared in the literature and adjust it for the quality factor.

Keywords: Inventory; Quality.

1. INTRODUCTION

The classical EOQ model, frequently referred to as Wilson’s Square Root Model, Wilson (1934), is based on a series of highly restrictive assumptions among which are constant deterministic demand, constant holding costs across time, and perfect quality of replenishment items. These assumptions limit the applicability of the Wilson’s model to the actual inventory systems that one encounters in organizations. In recognition of this fact, a vast number of researchers in the area of inventory management have developed more realistic inventory models by relaxing these
assumptions in various ways. Let us look at the assumption of constant holding cost. This assumption presumes that stock can be stored indefinitely to meet future demand. While this is true for many items, certain types of inventories can undergo changes in composition, potency, etc. which make them unfit for consumption. These inventories are, in fact, perishable. To account for this fact, researchers have developed a class of models that treat holding cost as a function of the amount of time that an item is held in inventory. Such models are known as perishable inventory models.

Nahmias (1982) presents a review of the early work in perishable inventory theory. He classifies perishability into two classes: fixed lifetime and random lifetime. The fixed classification includes those cases where the lifetime is known a priori to be a specified number of periods or length of time. Of greater interest to the present research is the random lifetime classification which includes those cases where the product lifetime is a random variable with a specified probability distribution. A special case in this classification is that where the lifetime of the product exhibits exponential decay. Nahmias (1982) reviews the work of a number of authors who have assumed deterministic demand along with exponential decay of items. In these cases the solution of an ordinary differential equation results in a relationship for inventory position over time which in turn is used to determine the optimal order quantity and optimal total cost. The case of stochastic demand presents a more difficult problem. In general, exponential decay problems with random demand are extremely difficult to handle when there is positive ordering lead-time.

Weiss (1982) considers a product facing a constant demand rate, constant replenishment lead-time, a fixed ordering cost, and holding cost that follows a relationship that is nonlinear in time.
Specifically the unit holding cost is an increasing function of time, t, and is given by \( H(t) = C_ht^{\gamma} \), where \( C_h \) and \( \gamma \geq 1 \) are constants. The author develops an Economic Order Quantity (EOQ) model that minimizes average combined ordering and holding costs over an infinite horizon. Ferguson et al (2007) is an approximation of the optimal order quantity for perishable goods such as produce and dairy products sold in small supermarkets that do not receive daily deliveries. Beyond this they apply the Weiss' model to data obtained from a national U.S. grocery chain. Using regression these data are fit to the nonlinear holding cost model. A series of numerical experiments are performed to test the performance of the model against the classical EOQ. The results of the study show Weiss' model delivers superior results under a series of robust scenarios. The improvement over the classical EOQ model is more significant for higher daily demand rate, lower holding cost, shorter lifetime, and a markdown policy with steeper discounts.

Ferguson et al (2007) implicitly assume that all items in a lot are of perfect quality, which is not always the case. Recognizing the practical importance of quality in operations, researchers have developed a number of inventory models that investigate the relationships among order quantity and quality. Rosenblatt and Lee (2009) investigated the effect of process quality on lot size in the classical economic manufacturing quantity (EMQ) model. Porteus (1986) introduced a modified EOQ model that indicates a significant relationship between quality and lot size. In both the above cases, demand is assumed to be deterministic. Moinzadeh and Lee (1987) investigated the effect of defective items on the order quantity and reorder point of a continuous-review inventory model with Poisson demand and constant lead-time. Paknejad et al (1995) extend this work to consider stochastic demand and constant lead-time in the continuous review \((s,Q)\) model. Nasri et al (2009)
develop a quality-adjusted EOQ model for the case where both backorders and stockouts are allowed.

Cheng (1991) develops a model that integrates quality considerations with the EPQ. The author assumes that unit production cost increases with increases in process capability and quality assurance expenses. Classical optimization results in closed forms for the optimal lot size and optimal expected fraction acceptable. The optimal lot size is intuitively appealing since it indicates an inverse relationship between lot size and process capability. It should be noted that a good survey of the early literature on integrating lot size and quality control policies is given in Goyal et al (1993).

In this early work the authors assume that the manufacturer operates a process that is in statistical control. That is, the process generates a known, constant proportion of defectives, $p$. Such an assumption induces a situation where the proportion of defective items follows a binomial distribution, and process quality, therefore, may be monitored by a proportion control chart. This assumption is also made in more recent work by Affisco et al (2009) for the case of the EOQ and Affisco et al (2002) for the case of the joint economic lot size model.

In all the above work the manufacturing process is assumed to be stable. Nasri, et al (2009) began to study the relationship between order quantity and quality for processes that have not yet achieved the state of statistical control for the cases of the EOQ with backorders, EMQ with backorders, and EOQ with two types of shortage, respectively, where the number of defectives produced by the manufacturing process is random rather than constant.
One of the assumptions in the above papers is that the holding cost is a linear function of time. In this paper we investigate the impact of the random quality of replenishment items on the optimal order quantity of the EOQ model with nonlinear holding cost as originally formulated by Weiss (1982) and later applied by Ferguson et al (2007) to inventory management of perishable goods.

2. REVIEW OF THE BASIC MODEL

The basic model considered in this paper is the classic EOQ with deterministic demand, constant setup cost, and nonlinear holding cost, developed by Weiss (1982) and applied by Ferguson et al (2007) as an approximation of the optimal order quantity of perishable goods. Assuming that the cumulative holding cost for one unit held during t interval of time is \( H(t) = C_h t^\gamma \), where \( C_h \) and \( \gamma \geq 1 \) are constants, the average inventory cost per unit time, \( AC_{basic}(Q) \), the resulting optimal lot size, \( Q_{basic}^* \), and the corresponding optimal average inventory cost per unit time, \( AC_{basic}^*(Q) \), are given by

\[
AC_{Weiss}(Q) = \frac{DK}{Q} + \frac{Q^\gamma C_h}{(\gamma + 1)D^{\gamma - 1}},
\]

(1)

\[
Q_{Weiss}^* = \left( \frac{1}{\gamma} + \frac{1}{\gamma} \right) \left( \frac{DK}{C_h} \right)^{\frac{1}{\gamma}},
\]

(2)

and

\[
AC_{Weiss}^*(Q) = \left( \frac{1}{\gamma} + \frac{1}{\gamma} \right) \left( \frac{DK^\gamma C_h}{\gamma} \right)^{\frac{1}{\gamma}},
\]

(3)

where

\[ D = \text{demand per unit time (in units)}, \]
K = setup cost per setup,

\( C_h \) = cumulative holding cost per unit per unit time,

\( AC(Q) \) = average inventory cost per unit time,

\( Q \) = lot size per order,

\( T = \frac{Q}{D} \) = cycle time, time between two successive orders.

Please note that equations (2) and (3) simply reduce to the corresponding results of the classical Wilson’s EOQ model when \( \gamma = 1 \).

### 3. THE QUALITY ADJUSTED MODEL

The basic model discussed before, assumes that all units produced by the vendor, in response to the purchaser’s order, are of acceptable quality. In what follows we extend the above model to the case where the proportion of defective items in each lot is a random variable. We assume that the purchaser inspects the entire lot upon arrival. We further assume that the purchaser’s inspection process is perfect, and that all rejected items are returned to the vendor at no cost to the purchaser. In addition, we assume that the inspection cost is paid by the vendor. Of course, it is likely that the vendor will recover some of these costs from the purchaser either directly or indirectly. Based on this scenario, we now adjust the EOQ model with nonlinear holding cost developed by Weiss (1982) for the quality factor as follows:

Let

\( \lambda \) = yield, being defined as the proportion of non-defective items in an order lot,

\( \lambda \in [\alpha, \beta] \) for \( 0 \leq \alpha < \beta \leq 1 \), a continuous random variable,
\( f(\lambda) \) = probability density function of \( \lambda \),

\[ E(.) = \text{mathematical expectation,} \]

\[ E(\lambda) = \mu = \text{mean of } \lambda, \]

\[ E(\lambda^{r+1}) = \mu_{r+1} = \int_{\alpha}^{\beta} \lambda^{r+1} f(\lambda) \, d\lambda, \]

\( y = \lambda Q \) = Number of non-defective items in a lot,

c\( (y) \) = Total cost per cycle given that there are \( y \) non-defective items in the lot of size \( Q \),

\( T = y/D \) = Cycle time, time between two successive placement of orders,

\[ EAC_{adjWeiss} (Q) = \text{Expected total cost per year.} \]

Consider one cycle of length \( T = y/D \). Note that the cumulative holding cost if one unit is kept in inventory during the cycle \([0,T] \) is \( C_k T^r = \int_{0}^{y/D} C_k \gamma r^{r-1} \, dt \). During the cycle, the inventory level changes with time according to \( I(t) = y - Dt \). Therefore, the total cost per cycle is

\[ c(y) = K + \int_{0}^{T} I(t) C_k t^{r-1} \, dt = K + \int_{0}^{y/D} (y - Dt) C_k \gamma t^{r-1} \, dt = K + \frac{(\lambda Q)^{r+1}}{(\gamma + 1)D^r} C_h \]

(5)

The average cycle time and cycle cost are

\[ E(T) = \frac{E(y)}{D} = \frac{E(\lambda Q)}{D} = \frac{Q}{D} \mu_t \]

(6)

and

\[ E(c) = K + \left( \frac{E(\lambda^{r+1})Q^{r+1}}{(\gamma + 1)D^r} \right) C_k = K + \frac{\mu_{r+1}Q^{r+1}}{(\gamma + 1)D^r} C_h \]

(7)

The expected average total annual cost is

\[ EAC_{adjWeiss} (Q) = \frac{E(c)}{E(T)} = \frac{DK}{\mu_t Q} + \left[ \frac{\mu_{r+1}Q^r}{(\gamma + 1)\mu_tD^{r-1}} \right] C_h \]

(8)
Using typical optimization techniques, we can easily find the optimal lot size, $Q^*_{\text{adjWeiss}}$, and the resulting optimal annual cost, $EAC^*_{\text{adjWeiss}}$, as follows:

$$Q^*_{\text{adjWeiss}} = \sqrt[\gamma+1]{\frac{\left(1 + \frac{1}{\gamma}\right)D^\gamma K}{C_h \mu_{\gamma+1}}}$$

and

$$EAC^*_{\text{adjWeiss}} = \frac{1}{\mu_1} \sqrt[\gamma+1]{\left(1 + \frac{1}{\gamma}\right)^\gamma \mu_{\gamma+1} D K^\gamma C_h}$$

Please note that when quality is perfect, that is $\lambda=1$, equations (9) and (10) simply reduce to the corresponding results in Weiss (1982), which were applied by Ferguson et al (2007) and subsequently reported in Nasri et al (2012). In addition, when quality is perfect and $\gamma=1$, the quality adjusted results of this paper lead to the corresponding results of the traditional EOQ model, originally developed by Wilson (1934).

**4. CONCLUSION**

This paper presents an extension of the EOQ model with nonlinear holding cost to account for imperfect quality of ordered items. Specifically, the paper assumes that yield, defined as the proportion of nonconforming items in each ordered lot, is a continuous random variable with known probability density function. Based on this assumption, the paper adjusts an EOQ model with nonlinear holding cost for the quality factor. Explicit results for the optimal order quantity and total cost are given for perishable inventory items with imperfect replenishment quality.
REFERENCES


